Finding Roots by "Open" Methods

The differences between "open" and "closed" methods

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closed	open
uses a bounded interval	not restricted to interval
(usually) converges slowly	(usually) converges quickly
always finds a root (if it exists)	may not find a root (if it exists)

Newton/Raphson method

This method uses not only values of a function **f(x)**, but also values of its derivative **f'(x)**. If you don't know the derivative, you can't use it.

The graphical approach to the method may be described as "follow the slope down to zero"; see your textbook for an illustration.

One can also use the Taylor series to derive Newton's method. The problem is: given

- a starting point **x1**
- a function **f(x)**
- the function's derivative **f'(x)**

how can we find a root of the function -- that is, a place **xr** where **f(xr) = 0**?

Suppose we expand the function around the starting point, using the Taylor series:

$$f(x) = f(x1) + f'(x1) * (x - x1)$$

Now, we want to find the spot where **f(x) = 0**, so let's plug that into the equation and solve for **x**.

$$0 = f(x1) + f'(x1) * (x - x1)$$

$$-f(x1) = f'(x1) * (x - x1)$$

$$\frac{-f(x1)}{f(x1)} = x - x1$$

$$x = x1 - \frac{f(x1)}{f'(x1)}$$

So, given a first point **x1**, one can calculate a new guess **x2** which -- we hope -- is closer to the root. Iterating a number of times might move us very close to the root.

But there is **no guarantee** that this method will find the root. The method often does, but it can fail, or take a very large number of iterations, if the function in question has a slope which is zero, or close to zero, near the location of the root. It can also fail if the second derivative of the function is zero near the root.

Example of Newton's method

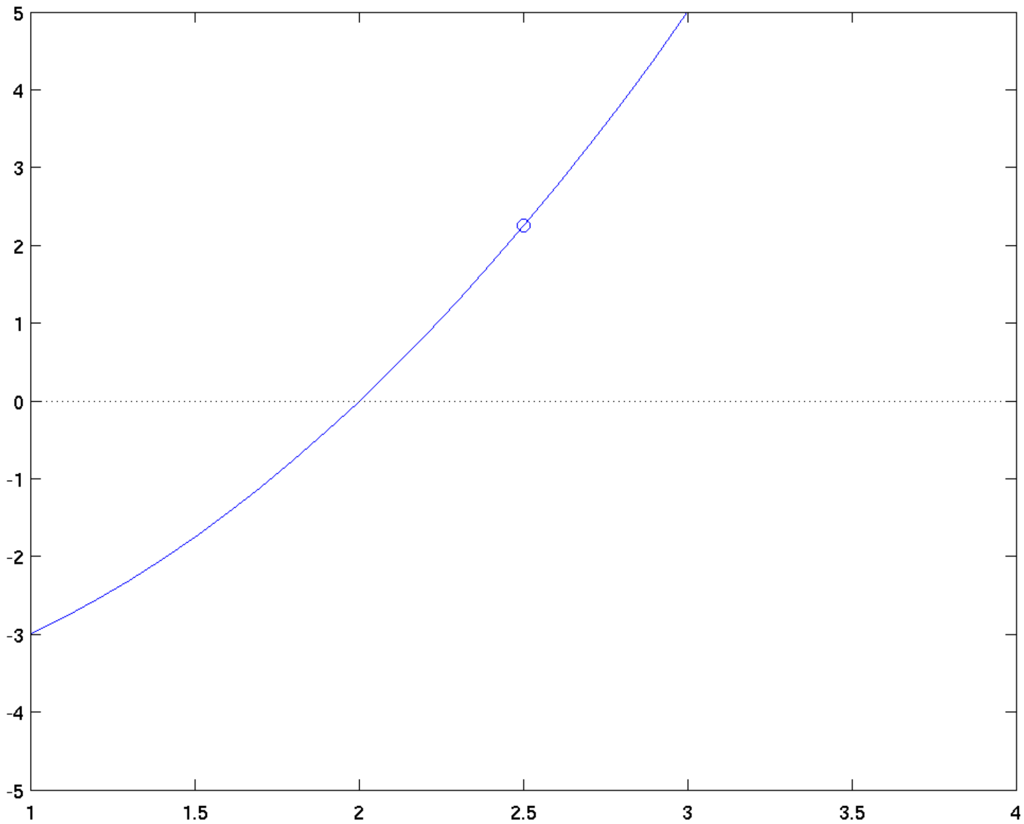
Let's look at a specific example of Newton's method:

find a root of the equation $y = x^2 - 4$ on interval [0, 5]

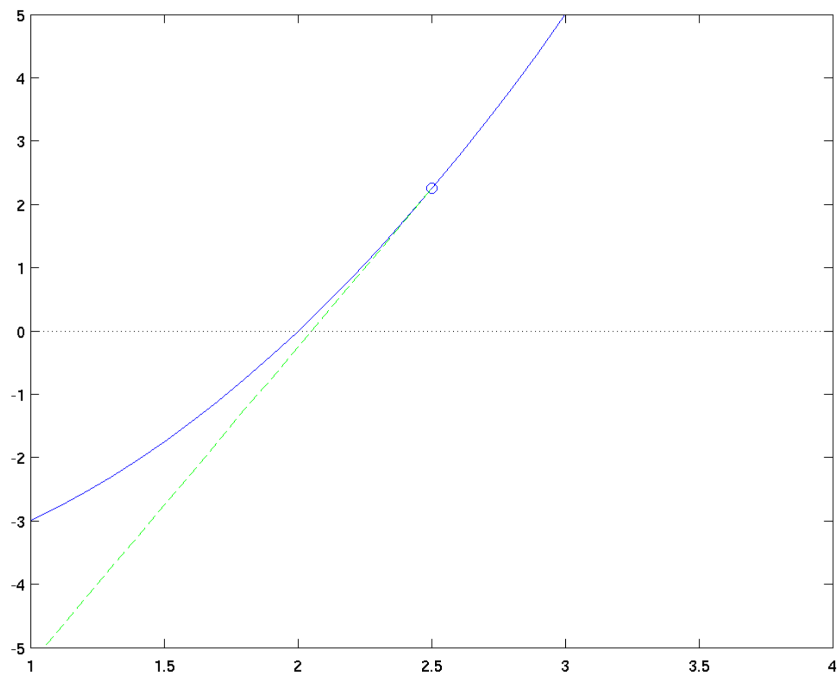
stop when relative fractional change is 1e-5

The exact root is 2, of course. How quickly can we find it, to within the given termination criterion?

The first step is to pick a starting point -- why not try halfway between the endpoints, at **x1 = 2.5**? The function has a value of **f(x1) = 2.25** there ...



Now, we calculate the derivative of the function at **$x1 = 2.5$**, which is **$f'(x1) = 5.0$.** So, we draw a line with slope 5 downwards from our current point, and follow it to the x-axis.

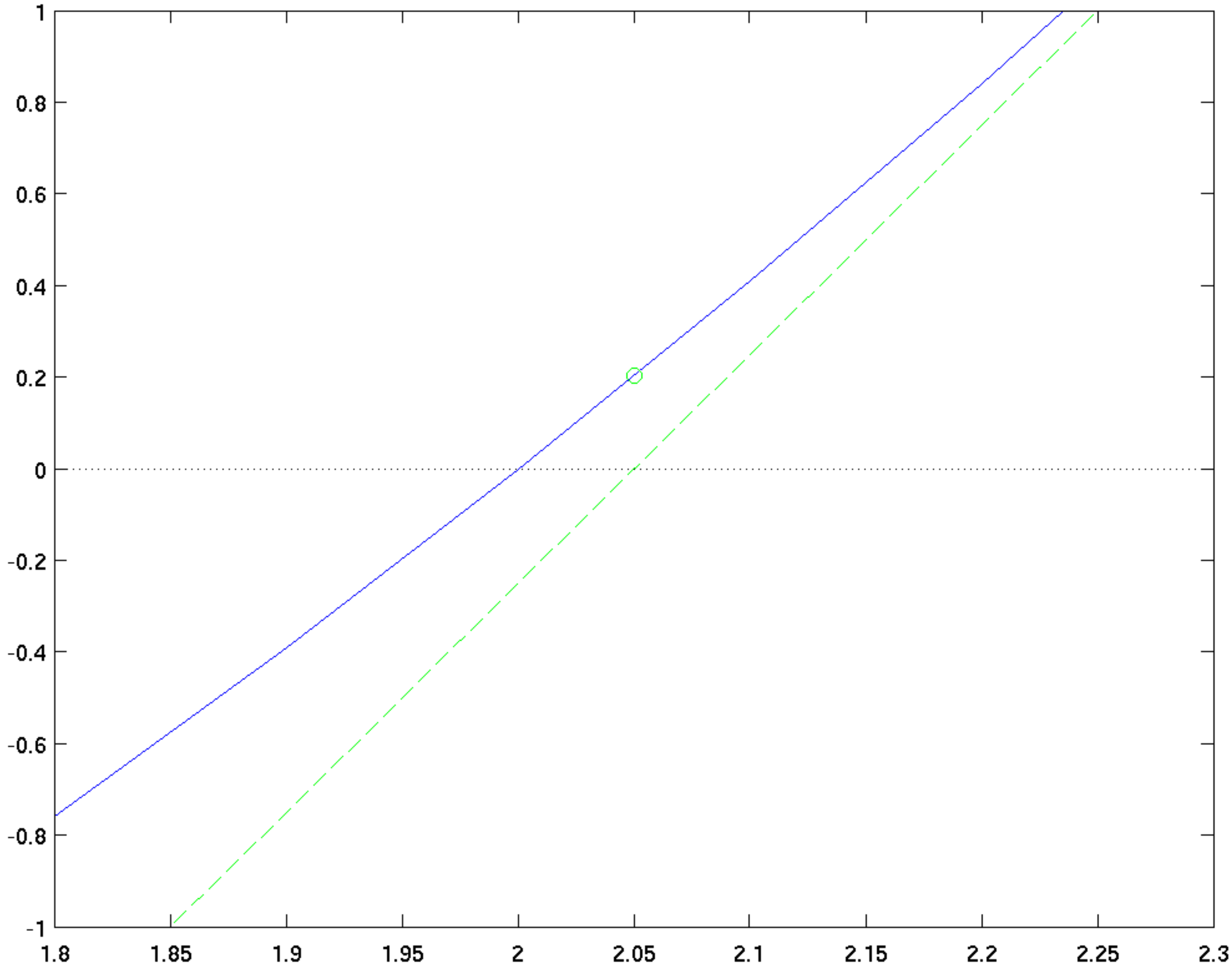


Where does this line intercept the x-axis? At the point given by

$$x2 = x1 - \frac{f(x1)}{f'(x1)}$$

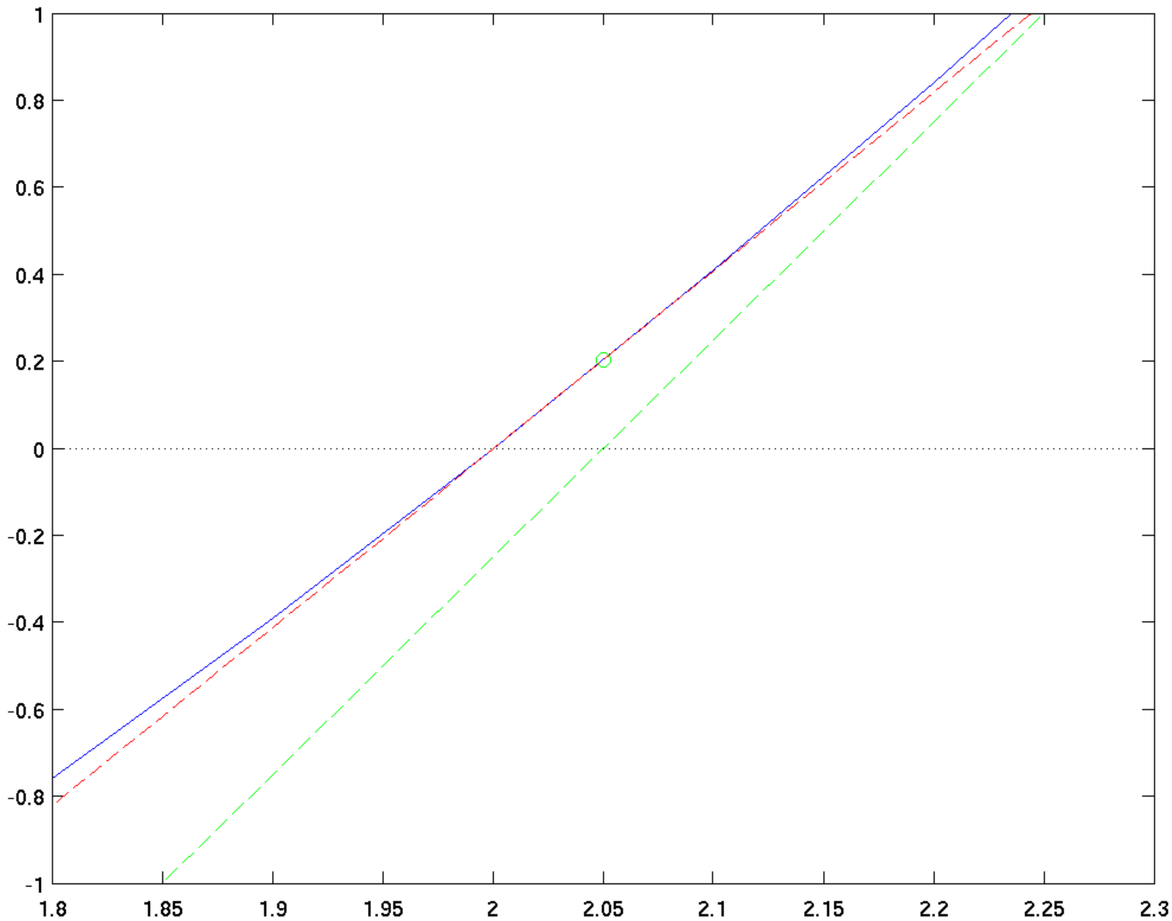
which turns out to be our next guess at the root: **x2 = 2.05**

We are now much closer to the root than we were at the start -- hooray! Newton's method sure is fast, when it works. Let's zoom in and repeat the process. We evaluate our function at the new position, finding **f(x2) = 0.2025.**



We also evaluate the derivative of the function at this point, which yields a new slope: **f'(x2) = 4.1.** Following this new slope down to the x-axis, we see that it intersects at

```
              f(x2)
x3   =   x2  -  ------
              f'(x2)
```



Wow. This third estimate of the root, x3 = 2.00061, is really, really close to the actual root (which is 2, of course). In just two steps, Newton's method has done very well.

You can watch the progress of Newton's method in these movies:

- Movie with fixed limits (slow version)
- Movie with fixed limits (fast version)
- Movie which zooms in after each step (slow version)
- Movie which zooms in after each step (fast version)

Here's a table showing the method in action:

```
       current       value at     value of deriv    fractional
 iter   guess     current guess  at current guess     change
-----------------------------------------------------------------
  0  2.500000e+00  2.2500e+00     5.0000e+00     2.1951e-01
```

1	2.050000e+00	2.0250e-01	4.1000e+00	2.4688e-02
2	2.000610e+00	2.4394e-03	4.0012e+00	3.0483e-04
3	2.000000e+00	3.7169e-07	4.0000e+00	4.6461e-08

The result is 2.00000000000000, which is indistinguishable from the true root of 2.

Secant method

In order to use Newton's method, we need to be able to calculate the derivative of a function at some point: **f'(x1).** Sometimes you don't know the derivative; what can you do then?

What you can do is **estimate** the derivative by looking at the change in the function near **x1**: pick some other point **x2** close to **x1**, and estimate the derivative as

```
                          f(x1) - f(x2)
derivative at x1  is approx  D =   -------------
                            x1  -  x2
```

With that approximation in hand, you can then apply the same method to guess the point at which the function equals zero.

```
              f(x1)
x_new   =   x1  -  ------
                D
```

After finding **x_new**, one can replace

```
x1 = x_new
x2 = x1
```

and make another iteration, and so forth.

The secant method therefore avoids the need for the first derivative, but it does require the user to pick a "nearby" point in order to estimate the slope numerically. Picking a "nearby" point which is too far, or too near, the first one, can lead to trouble. The secant method, just like Newton's method, is vulnerable to slopes which are very close to zero: they can cause the program to extrapolate far from the true root.

Source: http://spiff.rit.edu/classes/phys317/lectures/open_root/open_root.html