Finding Roots by "Closed" Methods

One general approach to finding roots is via so-called "closed" methods.

Closed methods

A **closed** method is one which starts with an interval, inside of which you know there **must** be a root. At each step, the method continues to produce intervals which**must** contain the root, and those intervals steadily (if slowly) shrink. These methods are guaranteed to find a root within the interval, as long as the function is well-behaved.

Open methods can be much faster, but they give up this certainty.

The two closed methods discussed here also require only that you be able to evaluate the function at some point; they do not require that you also be able to evaluate the function's derivative.

The Bisection Technique

Given some function **f(x)** which is real and continuous, and given some interval **[x1, x2]** inside of which you want to check for a root, do the following:

1. Calculate **f(x1)**
2. Calculate **f(x2)**
3. Check to see if they have the same sign. If so, there may be no root between them. Quit
4. If they have opposite signs, then
    a. Pick a point, **x_new**, halfway in between the two ends
    b. Look at the sign of **f(x_new)**
        I. If it's the same as **f(x1)**, the root lies between **x_new** and **x2**; so set **x1 = x_new**
        II. If it's the same as **f(x2)**, the root lies between **x1** and **x_new**; so set **x2 = x_new**
        III. If **f(x_new) = 0**, then you're done
    c. Go to step "a"

Good things about the bisection method:

- it's certain: the root MUST be inside the range
- it ALWAYS converges
- each iteration improves the estimate by a known about (how much?)
- only requires evaluation of function, not derivative

Bad things about the bisection method:

- may converge more slowly than some other methods

Example of Bisection method

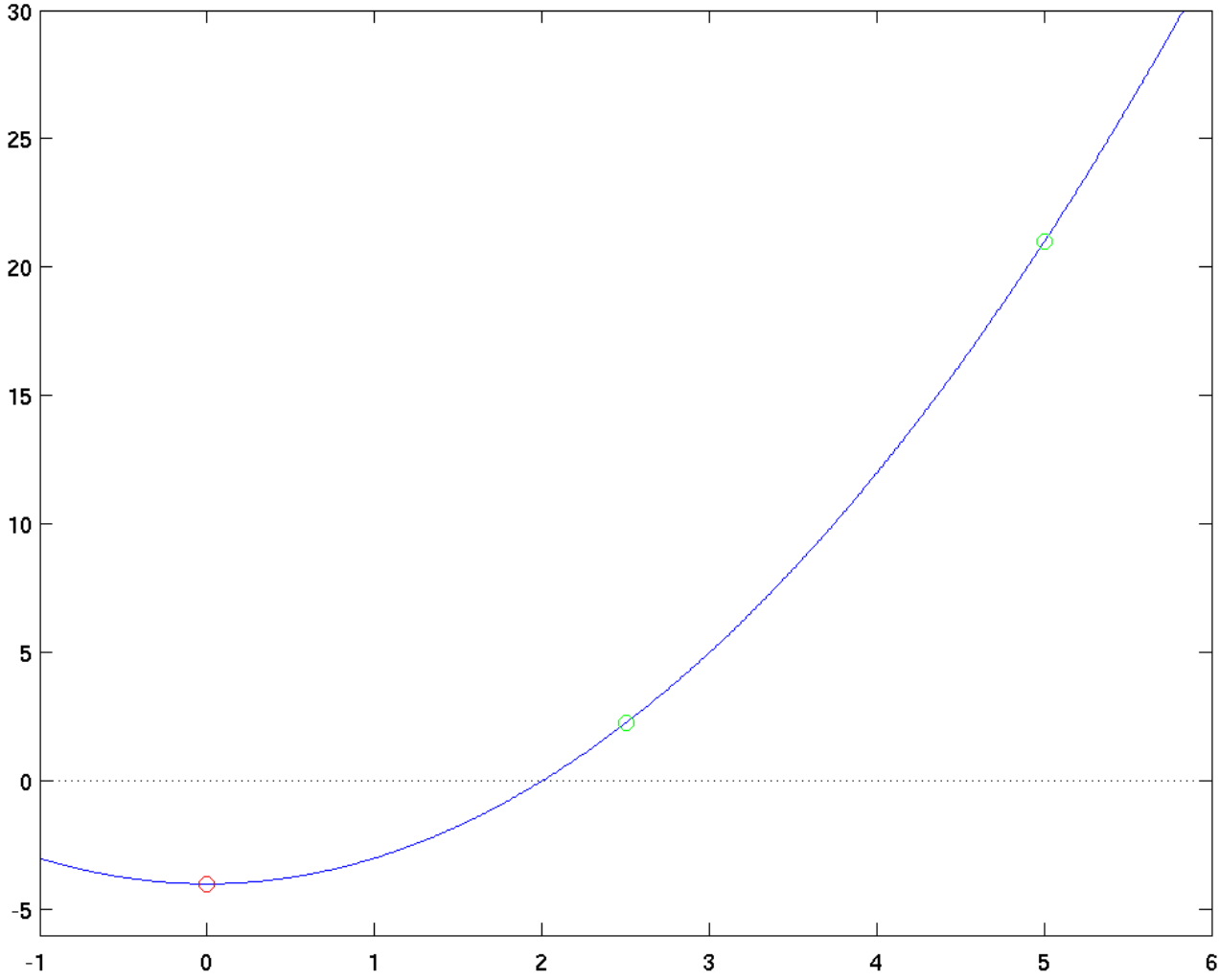
Let's look at a specific example of the bisection technique:

find a root of the equation y = x^2 - 4 on interval [0, 5]

stop when relative fractional change is 1e-5

The exact root is 2, of course. How quickly can we find it, to within the given termination criterion?

Let's watch the method in action, one step at a time. First, we need to calculate the value of the function at the bottom of the interval (**x1 = 0**), and the top of the interval (**x2 = 5**). We also make our first guess: the midpoint of the interval, **x_new = 2.5**, and evaluate the function there, too.

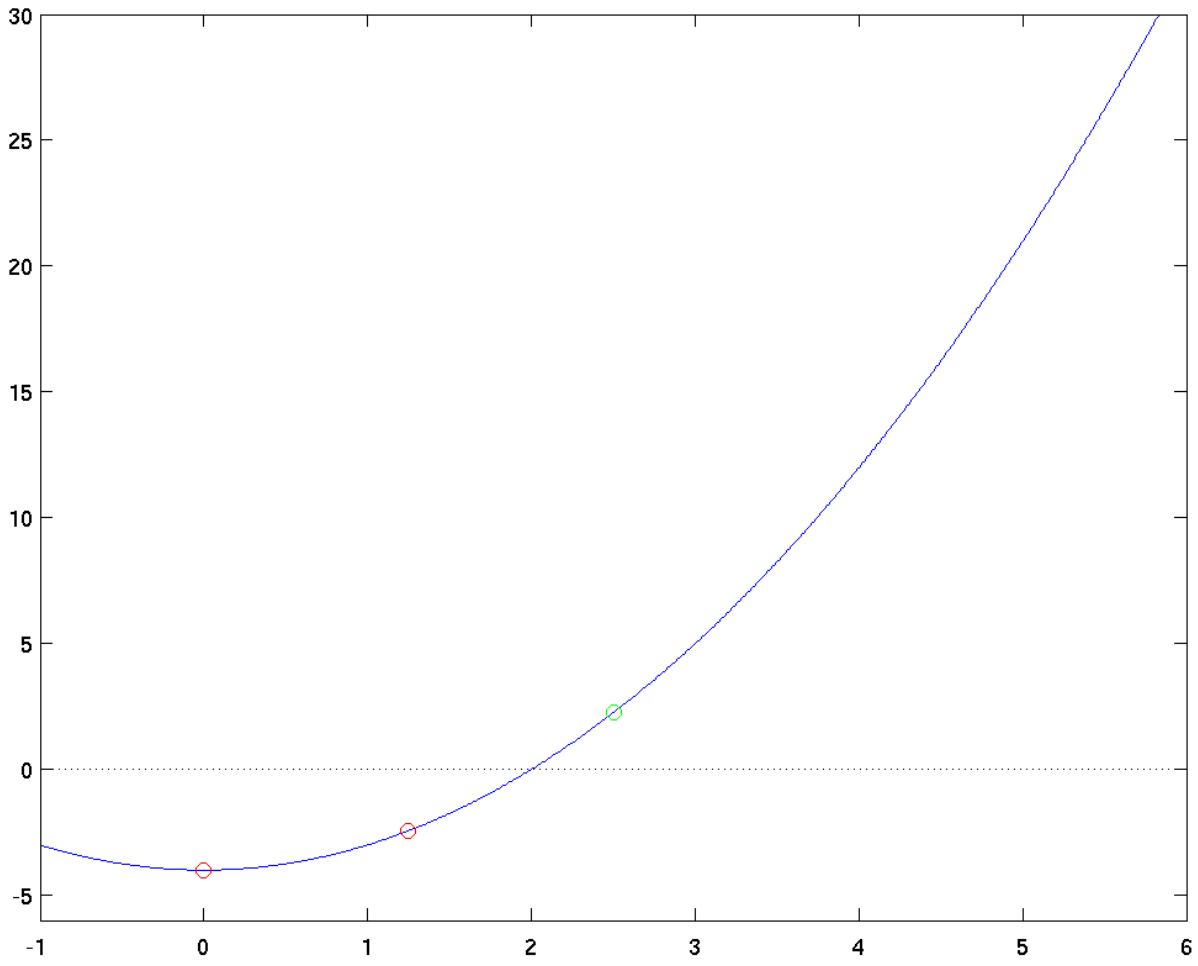


Now, looking at the sign of the function at each of these points, we see that

- the sign is negative at **x1**
- the sign is positive at **x_new**
- the sign is positive at **x2**

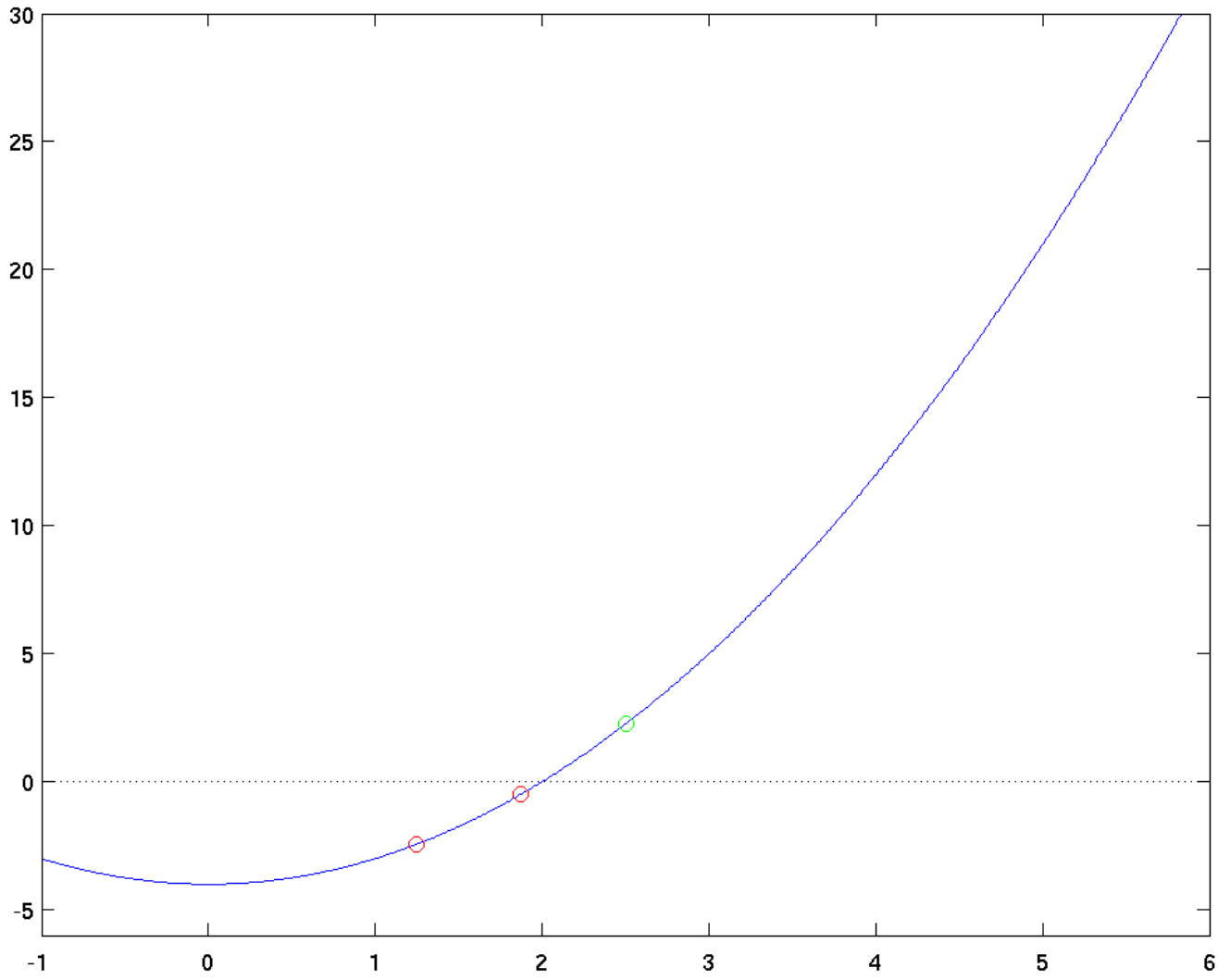
The root must lie somewhere between **x1** and **x_new**, because the function MUST change sign at a root. So, we leave **x1** = **0**, but move **x2** to the current midpoint. Our interval is now [0, 2.5].

We now repeat the bisection's basic step: find the midpoint of the current interval (**x_new** = **1.25**), and look at the sign of the function at the endpoints and the midpoint.



This time, the function changes value between the midpoint **x_new = 1.25** and the upper bound **x2 = 2.5**. That means that we'll discard the old lower bound, and re-set it so that **x1 = 1.25** for the next step. Our interval is now [1.25, 2.5].

Going back to the top of the loop, we find the midpoint **x_new = 1.875**, and evaluate the function here and at the endpoints.



It looks like we'll have to move the lower bound again in the next step ...

You can watch the progress of the bisection method in these movies:

- Movie with fixed limits (slow version)
- Movie with fixed limits (fast version)
- Movie which zooms in after each step (it only shows the first 4 iterations, but you'll get the idea)

Here's a table showing the method in action:

iter	lower bound	value at lower bound	upper bound	value at upper bound	fractional change
iter 0	0.0000000e+00	-4.0000e+00	5.0000000e+00	2.1000e+01	1.0000e+00
iter 1	0.0000000e+00	-4.0000e+00	2.5000000e+00	2.2500e+00	1.0000e+00
iter 2	1.2500000e+00	-2.4375e+00	2.5000000e+00	2.2500e+00	3.3333e-01
iter 3	1.8750000e+00	-4.8438e-01	2.5000000e+00	2.2500e+00	1.4286e-01
iter 4	1.8750000e+00	-4.8438e-01	2.1875000e+00	7.8516e-01	7.6923e-02
iter 5	1.8750000e+00	-4.8438e-01	2.0312500e+00	1.2598e-01	4.0000e-02
iter 6	1.9531250e+00	-1.8530e-01	2.0312500e+00	1.2598e-01	1.9608e-02
iter 7	1.9921875e+00	-3.1189e-02	2.0312500e+00	1.2598e-01	9.7087e-03
iter 8	1.9921875e+00	-3.1189e-02	2.0117188e+00	4.7012e-02	4.8780e-03
iter 9	1.9921875e+00	-3.1189e-02	2.0019531e+00	7.8163e-03	2.4450e-03
iter 10	1.9970703e+00	-1.1710e-02	2.0019531e+00	7.8163e-03	1.2210e-03
iter 11	1.9995117e+00	-1.9529e-03	2.0019531e+00	7.8163e-03	6.1013e-04
iter 12	1.9995117e+00	-1.9529e-03	2.0007324e+00	2.9302e-03	3.0516e-04
iter 13	1.9995117e+00	-1.9529e-03	2.0001221e+00	4.8830e-04	1.5260e-04
iter 14	1.9998169e+00	-7.3239e-04	2.0001221e+00	4.8830e-04	7.6295e-05
iter 15	1.9999695e+00	-1.2207e-04	2.0001221e+00	4.8830e-04	3.8146e-05
iter 16	1.9999695e+00	-1.2207e-04	2.0000458e+00	1.8311e-04	1.9073e-05
iter 17	1.9999695e+00	-1.2207e-04	2.0000076e+00	3.0518e-05	9.5368e-06

The result is 1.99998855590820, which is indeed close to 2.0.

Termination conditions

Now, if you try to implement the method shown above, you may discover that your program never stops running -- or runs for a long, long time. Ooops. The algorithm has only one **termination condition**: if you find the EXACT root, then quit; otherwise, keep going. In most cases, the computer can't represent the EXACT root, and so you may never find it.

You should always include a **termination condition** in a root-finding function ... and you should always think about termination conditions in any iterative method. There

are a number of possibilities, but all of them require that you pick some small value **epsilon**.

1. if the absolute value of the function falls below **epsilon**, quit. We're looking for a root, a place where **f(x)** equals zero, so
2. | f(x) | < epsilon

   might make sense; we're probably close ...

3. if the latest value of **x** is within **epsilon** of the true root value, then quit:
4. | x_new - x_true | < epsilon

   Unfortunately, we don't know the true root **x_true**. Oops.

5. if the latest value of **x** is only a teensy bit different from the previous value, then quit:
6. | x_new - x_prev | < epsilon

   This is reasonable ...

7. if the fractional change in **x** is very small, then quit:
8. | x_new - x_prev |
9. | ---------- | < epsilon
10. | x_new |

    This is one of the most commonly used criteria. Note that bad things can happen if the root is near zero, because we will then be dividing by a very small number.

False Position Technique

The basic idea here is to use an estimate for the new value of the root which is better than the midpoint of the range. As before, it starts with a boundary for which the function is positive at one end and negative at the other. The method assumes that the function changes linearly from one end to the other, and calculates the value of**x_new** at which that linear approximation crosses zero.

An algorithm for this method looks almost identical to the one above for the Bisection method, but in Step 4a, instead of

4. If they have opposite signs, then
    a. Pick a point, **x_new**, halfway in between the two ends

you make a linear fit to the function between the two ends, and find the spot where that fit crosses zero:

4. If they have opposite signs, then
    a. Pick a point, **x_new**, at which a linear fit between **f(x1)** and **f(x2)** equals zero

You can interpolate from either end; the textbook works out the calculation of **x_new** working backwards from the upper end **x2**.

Good things about the false position method

- it ALWAYS converges
- it usually converges more quickly than the Bisection method
- only requires evaluation of function, not derivative

Bad things about the bisection method:

- each iteration improves the estimate by an unknown amount
- may converge very slowly for some functions

Source: http://spiff.rit.edu/classes/phys317/lectures/closed_root/closed_root.html