Data & Procedure Hashing

While trees are a useful data structure for implementing the Set ADT, practitioners tend to use the *hash table* instead. A tree maintain its elements in order, which is useful in some cases. Hash tables do not maintain any order; this flexibility, as we shall see, results in an opportunity for improved speed.

Before investigating the technique of hashing, though, we'll first examine the important Map ADT, another ADT for which both trees and hashing are useful.

6.1. The Map ADT

The Map ADT is designed for cases where a program wants to remember associations between objects — that is, where the program wants to maintain a *mapping* from some collection of objects to another set of objects.

6.1.1. Lookup tables

Before we look at the general Map ADT, we first consider a simple structure called a lookup table. Here, we use an array to associate values with a small range of integers.

An example involving a lookup table would be a program to compute the *mode* of a set of test scores — that is, the score that occurs most often within a list. There are several ways of computing the mode without using a lookup table. The simplest technique is to go through each element, and for each element count how many other times that element occurs in the list. We return the element with the maximum count. This technique, with two nested loops, each iterating through all elements of the list, takes $O(n^2)$ time for n test scores.

We can improve this dramatically, still without using a lookup table, by sorting the list: After sorting, all repeated scores will be adjacent to one another, and we have only to go through the sorted list to find the longest sequence. Sorting the list takes $O(n \log n)$ time, and finding the longest sequence takes $O(n)$ time; thus the total time for this algorithm is $O(n \log n)$.

Figure 6.1: Finding the mode of an array using a lookup table.

```
public static int findMode(int[] scores) {
    // compute # occurrences of each score
    int[] count = new int[101];
    for(int i = 0; i < count.length; i++) count[i] = 0;
    for(int i = 0; i < scores.length; i++) count[scores[i]]++;

    // now find maximum # occurrences
    int max = 0;
    for(int i = 1; i < count.length; i++) {
        if(count[i] > count[max]) max = i;
    }
    return max;
}
```

But we can do better using a lookup table. Since the test scores will fall within a restricted range, we can create an array that will track the number of occurrences of each test score. After determining the proper values for this array, we can easily find which element has the largest value. Figure 6.1 illustrates this technique. The first and last loops of Figure 6.1 both take $O(1)$ time relative to n: Both have at most 101 iterations, each iteration taking $O(1)$ time, for a total of $O(101 \cdot 1) = O(1)$ time. The middle loop, though, has n iterations, so it takes $O(n)$ time.

But suppose we want to be able to find the mode from a list of strings? Or of points? Or of cities? We might hope to use the lookup table again, but we couldn't because a lookup table relies on the elements of the list being used as indices into an array, and only `int`s can be used as array indices. What we want is a different kind of array — one in which any type of data can be used as an index. This, in fact, is precisely the purpose of the Map ADT that we'll now examine.

6.1.2. The Map ADT

The two abstract data types we have seen thus far, the List ADT and the Set ADT, operate on collections of elements — an ordered collection for the List ADT, an unordered collection for the Set ADT. The Map ADT, however, does not deal with a collection of elements: It corresponds to a *mapping* from some values (called *keys*) to other values. This is appropriate when an algorithm wants to associate values together. In our example of computing a mode, we want to associate integer counts with values found in the source list; the source values will be the keys to the map.

The Map ADT contains the following operations.

- `get(k)` to retrieve the value associated with the key `k`.
- `put(k, x)` to associate the value `x` with the key `k`.
- `remove(k)` to remove any value associated with the key `k`.
- `containsKey(k)` to query whether `k` has a value associated with it.

- `keySet` to get the set of keys with associated values.
- `size` to get the number of keys with associated values.

As you would expect, java.util contains a Map interface corresponding to the Map ADT. In contrast to the generic classes and interfaces we have seen so far, which have all had one generic parameter (like the `<E>` of `List<E>`), the Map interface has two generic parameters, `K` and `V`.

```
public interface Map<K,V> {
    public V get(K key);
    public V put(K key, V value);
    public V remove(K key);
    public boolean containsKey(K key);
    public Set<K> keySet();
    public int size();
}
```

6.1.3. Using a Map

In addition to the Map interface, the java.util package also includes a TreeMap class implementing it. The TreeMap works similarly to a TreeSet, but each tree node in a TreeMap contains both a key and a value. The nodes are kept in the tree based on their keys' order.

We can see how to use the Map interface, and the TreeMap implementation of that interface, by going back to our example of finding a mode. Suppose we want to find the most commonly occurring string within an array of names. According to the logic of the algorithm earlier, we would want a mapping from strings to integers. And so we'd use a `Map<String,Integer>`, as illustrated in Figure 6.2. This isn't a straightforward line-by-line translation, but but it does follow the same basic algorithm.

Figure 6.2: Finding the mode of an array using a hash table.

```
public static String findMode(String[] names) {
    // compute # occurrences of each score
    Map<String,Integer> count = new TreeMap<String,Integer>();
    for(int i = 0; i < names.length; i++) {
        if(count.containsKey(names[i])) {
            int oldCount = count.get(names[i]).intValue();
            count.put(names[i], new Integer(oldCount + 1));
        } else { // first occurrence found
```

```
            count.put(names[i], new Integer(1));
        }
    }

    // now find maximum # occurrences
    String mode = null;
    int modeCount = 0;
    for(Iterator<String> it = count.keySet().iterator(); it.hasNext(); ) {
        String name = it.next();
        int nameCount = count.get(name).intValue();
        if(nameCount > modeCount) {
            mode = name;
            modeCount = nameCount;
        }
    }
    return mode;
}
```

For the TreeMap class, the `get` and `put` operations each take $O(\log n)$ time. Thus, each iteration of the first loop takes $O(\log n)$ time; there are n iterations, for a total of$O(n \log n)$ time for the first loop. Likewise, the second loop takes $O(\log n)$ time per iteration, and there are at most n iterations, for $O(n \log n)$ time. Thus, overall, this approach takes $O(n \log n)$ time.

6.2. Hash tables

While a TreeMap implements the `get` and `put` methods in $O(\log n)$ time, the equivalent operations for a lookup table take only $O(1)$ time. Unfortunately, the lookup table is applicable only when the keys happen to be small integers. The idea of *hashing* is to figure out a way to leverage the lookup table's performance for other types of keys. This leads to the HashMap class, an alternative implementation of the Map interface. In fact, because HashMaps are faster (and, because they don't involve the Comparable interface, easier to use), programmers use HashMap much more often than TreeMap in practice. The only advantage of TreeMaps is that they keep their keys in order, but this is not usually very important.

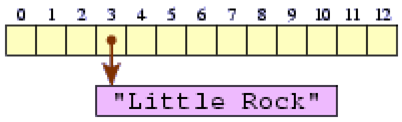
6.2.1. Concepts

Hashing relies on assuming a hash function, which maps keys to integers that can then be used as indices into a lookup table. Ideally, all different keys would hash to different integers, while any two equal keys would hash to the same integer.

Suppose that we want a map between U.S. states' postal abbreviations (Strings) and the states' capitals (also Strings). Suppose, also, that we have an ideal hash function that maps each state abbreviation to the position of the state when states are listed in alphabetic order by their full name; thus, Alabama (`AL`) maps to 0, Alaska (`AK`) maps to 1, etc.

We might tell a HashMap variable `map` to map Arkansas to Little Rock. The HashMap would first query the hash function for the code for `AR`, which would be 3 in this case. Then it would deposit the associated value into an array.

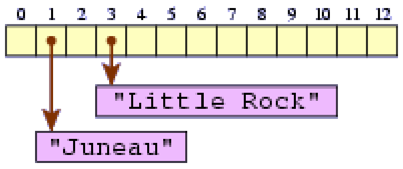
```
map.put("AR", "Little Rock");
```



You can see that the HashMap is simply a lookup table with an added step of applying the hash function to any key to determine the index within the lookup table.

If later we decide to map Alaska to Juneau, we can do that.

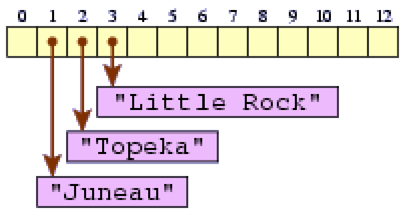
```
map.put("AK", "Juneau");
```



Now, given a requested to retrieve Arkansas's state capital (`map.get("AR")`), we would query the hash function for Arkansas's hash code (still 3), go to that entry of the table, and return the string found there (`Little Rock`).

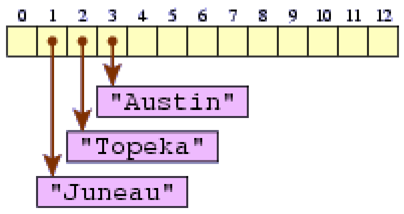
We can continue putting states' capitals into the table, but what do we do when we get to states whose hash codes fall outside the array's bounds? For example, what about Kansas (`KS`), whose hash code is 15, even though the array has only 13 entries? We'll map such a code into the desired range by taking the remainder when the code is divided by the array length; the remainder will be at least zero but less than the array length, and so it will be a valid array index. For the Kansas key, we'd divide 15 by 13, getting a remainder of 2, and so we'd place Topeka at index 2 in the table.

```
map.put("KS", "Topeka");
```



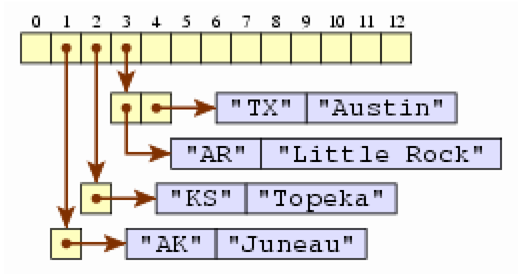
There are still other problems to address, though. Suppose we decide to map Texas (`TX`) to Austin. Texas has an index of 42, whose remainder is 3 when divided by 13. So we'd place Austin at index 3.

```
map.put("TX", "Austin");
```



Note that this replaced Little Rock. Now, if we ask our map for Arkansas' capital (`map.get("AR")`), we'll get Austin.

What we have here, with different keys mapping to the same hash table entry, is called a collision. Collisions are not desirable, but they're unavoidable. To deal with them, we'll instead put a list at each node, called a bucket. In each bucket, we'll have *entries*, each containing a key that maps into that bucket and the value associated with that key.



In fact, collisions are quite common. Suppose we decided to hash people based on their birthday. There are 366 different possible hash values, and this hash function is a good one in that it distributes people relatively evenly across them. Even so, an analysis of probabilities says that there will likely be a collision by the time we reach 23 different people.

This phenomenon, in fact, is called the birthday paradox. It's not a real paradox, but it gets that name because the fact runs counter to many people's intuition. The same analysis underlying the birthday paradox leads to the conclusion that any mapping from one set to another, even if it is quite good, is bound to lead to some collisions.

Now, if we're asked to retrieve the capital associated with Texas, we'll go to the bucket at index 3, and step through the list until we find an entry with Texas as the key. Once found, then we'll return the value in that entry — Austin.

6.2.2. Implementing HashMap

In order to be able to implement a hash table, we must have some way to access the hash function. A hash function must return the same value whenever two instances are equal. Different classes have different definitions of equality between instances. (Think of the different notions of equality between String objects and between Integer objects.) Thus, there is no way to write a generic hash function that covers all possible types of keys.

Java's approach to this is to define two methods — `equals` and `hashCode` — in the Object class, which all classes ultimately extend.

```
public boolean equals(Object other) { ...
public int hashCode() { ...
```

The default implementations (as the methods are defined in Object) treat two objects as equal exactly when they are the same object — i.e., they are at the same location of memory. This matches the notion of equality used with the `==` operator: The `equals` method defined in the Object class returns **`true`** whenever **`this`** `== other`. A class might override Object's `equals` method to define a different notion of equality; the String class is an example, because two Strings should be regarded as equal whenever they represent the same sequence of letters, even if they lie at different locations in memory. But if it overrides `equals`, it should also override `hashCode` so that two objects that `equals` say are equal will also get equal hash codes. (The String class overrides both.)

The HashMap class will include a private instance variable `table`, which will be an array of HashEntries, each being the head of a list of all entries in the bucket. (The name *`table`* comes from the term *hash table*, which is a common name for this data structure.) Whenever we want to look up a key named *`key`*, we want to access the bucket (to be named *`bucket`*) corresponding to that key's hash code. This is easy enough.

```
bucket = table[key.hashCode() % table.length];
```

This uses the remainder operator (`%`) to get the remainder when the hash code is divided by the array's length. This does not exactly work, though, because the hash code may be negative, and Java specifies that the remainder operator, when applied to a negative number, computes a negative remainder (e.g., `(-8) % 3` yields −2). We can avoid this possibility by adding `table.length` to the remainder should it turn out to be negative.

```
int index = key.hashCode() % table.length;
if(index < 0) index += table.length;
bucket = table[index];
```

Within each bucket, we want to store objects that associate keys and values together. For this purpose, we'll use the simple HashEntry class.

```
class HashEntry<K,V> {
    private K key;
    private V value;
    private HashEntry<K,V> next;

    public HashEntry(K k, V v, HashEntry<K,V> n) {
        key = k; value = v; next = n;
    }

    public K              getKey()      { return key; }
    public V              getValue()    { return value; }
    public void           setValue(V v) { value = v; }
    public HashEntry<K,V> getNext()     { return next; }
    public void           setNext(HashEntry<K,V> v) { next = v; }
}
```

Note that the class has no `setKey` method. This is intentional: Once we insert an entry into a bucket, we should never change the key associated with it, because a different key would likely belong in another bucket instead.

With the HashEntry class defined, we can now talk about the proper type for a bucket: Each bucket is a HashEntry, which may itself be the head of a list of HashEntries. Thus, the instance variable `table` is an array of HashEntries.

```
private HashEntry<K,V>[] table;
```

We'll use this within our HashMap implementation found in Figure 6.3, which illustrates how the implementation built into java.util works.

Figure 6.3: The HashMap class.

```
public class HashMap<K,V> implements Map<K,V> {
    private HashEntry<K,V>[] table;
    private int curSize;

    public HashMap() {
        table = (HashEntry<K,V>[]) new HashEntry[13];
        curSize = 0;
    }

    private int getIndex(K key) {
        int ret = key.hashCode() % table.length;
        if(ret < 0) ret += table.length;
        return ret;
    }

    public int size() { return curSize; }

    public V get(K key) {
        for(HashEntry<K,V> n = table[getIndex(key)]; n != null; n = n.getNext
()) {
            if(n.getKey().equals(key)) return n.getValue();
        }
        return null;
    }

    public boolean containsKey(K key) {
        for(HashEntry<K,V> n = table[getIndex(key)]; n != null; n = n.getNext
()) {
            if(n.getKey().equals(key)) return true;
        }
        return false;
    }

    public V put(K key, V value) {
        int index = getIndex(key);
        for(HashEntry<K,V> n = table[index]; n != null; n = n.getNext()) {
            if(entry.getKey().equals(key)) {
                V old = entry.getValue();
                entry.setValue(value);
                return old;
            }
        }
        table[index] = new HashEntry<K,V>(key, value, table[index]);
        curSize++;
        return null;
    }

    public V remove(K key) {
        int index = getIndex(key);
        HashEntry<K,V> prev = null;
        for(HashEntry<K,V> n = table[index]; n != null; n = n.getNext()) {
            if(n.getKey().equals(key)) {
                if(prev == null) table[index] = n.getNext();
                else             prev.setNext(n.getNext());
                curSize--;
                return n.getValue();
            }
            prev = n;
        }
        return null;
    }

    // keySet and iterator methods omitted
}
```

6.2.3. Performance

Hast fast is a HashMap? Its performance depends heavily on how big the buckets are. If the hash function is reasonably good, then all buckets will have roughly the same size. Assuming this, each bucket used in our Figure 6.3 implementation will have roughly n / 13 entries, since that implementation uses only 13 buckets. The `get`and `put` methods both rely on iterating through the relevant bucket's entries; since they have close to $n/13 = O(n)$ entries, this means that the methods take $O(n)$ time. Such performance is unacceptable.

The HashMap that is actually implemented in the java.util package will grow its table we add more entries to it, just as an ArrayList grows as it receives more values. In particular, when the number of entries reaches 75% of the number of buckets, the next call to `put` will create a new array for `table` approximately twice as long as before, and it will hash all the previous entries into their newly appropriate buckets. As with ArrayLists, this will happen infrequently enough that the $O(n)$ time taken by this doubling procedure adds only $O(1)$ time to `put` on average. The modified `put` method is in Figure 6.4.

Figure 6.4: When the table is 75% full, `put` doubles the table length.

```
public V put(K key, V value) {
    int index = getIndex(key);
    for(HashEntry<K,V> n = table[index]; n != null; n = n.getNext()) {
        if(entry.getKey().equals(key)) {
            V old = entry.getValue();
            entry.setValue(value);
            return old;
        }
    }
    if(curSize >= table.length * 3 / 4) { // double table length
        HashEntry<K,V>[] oldTable = table;
        table = (HashEntry<K,V>[]) new HashEntry[table.length * 2];
        index = getIndex(key);
        for(int i = 0; i < oldTable.length; i++) {
            HashEntry<K,V> n = oldTable[i];
            while(n != null) {
                HashEntry<K,V> next = n.getNext();
                int ni = getIndex(n.getKey());
                n.setNext(table[ni]);
                table[ni] = n;
                n = next;
            }
        }
    }
    table[index] = new HashEntry<K,V>(key, value, table[index]);
    curSize++;
    return null;
}
```

(The 75% amount is called the load factor. Java's HashMap includes constructors that allow you to customize how large the initial table is and what the load factor is, for programmers who need to tune the performance of their programs. The defaults usually work well enough in practice, though.)

With this doubling operation in place to ensure that the average bucket has $O(1)$ length, both `get` and `put` (and, for that matter, `remove` and `containsKey`) take $O(1)$ time, assuming that the hash function being used distributes keys approximately equally across the buckets. The assumption of equal distribution is important, here: In the worst case, a poor hash function might end up assigning all keys to the same bucket. In this case, even though the average bucket may have $O(1)$ length, the only bucket ever used will have $O(n)$ entries, and so the methods take $O(n)$ time. In a moment, we'll investigate how to define good hash functions to avoid such performance.

Given a good hash function, though, HashMap provides $O(1)$ performance for `get` and `put`, which is an improvement over TreeMap's $O(\log n)$ performance. So why would anybody ever use TreeMap instead? HashMaps do have one disadvantage: They don't maintain any order among the elements in the key set. Moreover, with the doubling operation, the order may change over time. Thus, TreeMap is suitable for cases where the keys should be maintained in a particular order, or where for some reason the order of the keys, though not important, should at least remain consistent. In practice, such cases are relatively rare, and thus HashMap is usually the better choice.

The java.util library also defines a HashSet class implementing the Set interface, which provides $O(1)$ performance for Set's `add`, `remove`, and `contains` operations, assuming a good hash function. This, too, is a marked improvement over TreeSet; but again, HashSet has the disadvantage that it doesn't maintain the values in any consistent order. In cases where the order is unimportant (and these happen more often than cases where order matters), HashSet is a better choice than TreeSet.

6.3. Hash functions

Until now, we have assumed that a good hash function was already implemented for us. The classes built into Java's libraries, indeed, already have good hash functions defined.

But when we define our own classes that we might possibly use as keys to hash tables, then we need to worry about whether our hash function is suitable. The hash function needs to satisfy two requirements.

- The hash code assigned to an object should be consistent over time. For example, we can't simply return a newly generated random number each time `hashCode`is called. If we had such a hash function, then keys would become lost in the hash table.

  One consequence of this is that the hash function should not depend on any data within the object that could possibly change in the future.

- The hash codes should be consistent with the behavior of the `equals` method. That is, if two objects are equal according to `equals`, then their hash codes should also be equal.

  This rule has an important consequence: If at any time you want to override Object's `equals` method, then you should also override its `hashCode` method if there is a reasonable chance that you or somebody else will want to use objects of the class as keys in a hash table.

In addition to these two required properties, there is also one property that is highly desirable but not really achievable.

- If two objects aren't equal according to `equals`, then their hash codes should not be equal.

  There are many situations where this property is impossible to satisfy: Consider the String class as an example. There are many more strings than there are valid**`int`**s, and so we could not possibly map every possible distinct string into a different **`int`**. Since the property is impossible to satisfy, the most we can hope for is that two unequal objects are likely to have unequal hash codes. (Fortunately, String's `hashCode` method has this property.)

As an example where you would want to override Object's `hashCode` method, consider a class to represent points with two integer coordinates.

```
public class Point {
    private int x;
    private int y;

    public Point(int x, int y) {
        this.x = x;
        this.y = y;
    }

    public int getX() { return x; }
    public int getY() { return y; }

    public boolean equals(Object other) {
        if(!(other instanceof Point)) return false;
        Point o = (Point) other;
        return o.x == this.x && o.y == this.y;
    }
}
```

In this case, we overrode `equals` because we want two points to be regarded as equal any time that their coordinates match. And whenever we override `equals`, we should override `hashCode` also.

So how can we define `hashCode` so that any two equal points receive the same hash code? One simple technique is to define the hash code be the point's *x*-coordinate.

```
public int hashCode() {
    return x;
}
```

This, however, means that any two points with the same *x*-coordinate get the same hash code, even if they have different *y*-coordinates. But it seems reasonably likely that in real-world scenarios we might end up with a HashSet or HashMap involving a cluster of Points, of which several would share the same *x*-coordinate. Thus, while this is a valid hash function, it is not a good one. In general, a hash function ought to take into account all the data that the `equals` method takes into account; here, the `equals` method relies in part of the points' *y*-coordinates, so we should probably include `y` in our computation for `hashCode`, too.

We can write such a hash function by simply adding the *x*- and *y*-coordinates.

```
public int hashCode() {
    return x + y;
}
```

This, though better, still has a shortcoming: Again, consider the possibility of a cluster of points. All points lying on a reverse diagonal (i.e., a line of slope −1) will share the same hash code. For example, points (4, 5) and (3, 6) receive the same hash code, even though they are still rather close.

We can improve the function by multiplying the *x*-coordinate by some coefficient.

```
public int hashCode() {
    return x * 31 + y;
}
```

Now, two points will receive the same hash code only if they lie on the same line with slope −31. With integer coordinates, such points would be relatively far apart: For example, the closest points to (4, 5) with the same hash code are (3, 36) and (5, −27). It seems unlikely that somebody would happen to have a set including many such points, so this is a reasonably good hash function. (Technically, due to issues with arithmetic going beyond the range of valid integers, there are other situations where two points receive the same hash code. Having two such points in the same set, though, is unlikely, and so we don't need to worry about it.)

This technique of computing hash codes — taking each relevant data member, multiplying them by very different coefficients, and adding the results together — is a common one, because it is both simple and it provides good results in practice. In fact, the `hashCode` method defined by Java's designers for String is another good example of this technique. One string is equal to another only if all of the characters of its string match the other; hence, its hash function ought really to incorporate information about all the characters of the string. Java's designers chose the function

$31^{k-1} s_0 + 31^{k-2} s_1 + \ldots + 31\, s_{k-2} + s_{k-1}$,

where k is the string's length, and s_i is the character at position i of the string.

With a bit of practice, defining a reasonably good hash function is easy, and that's all that's necessary for being able to use a hash table any time you want a Set or a Map. Hash tables constitute one of the most useful data structures in real computer programming: They are easy to use, efficient in their performance, and flexible in the types of objects they manipulate.

Source: http://www.toves.org/books/data/ch06-hash/index.html