Module 4

MULTI-RESOLUTION ANALYSIS

Lesson 10

Theory of Wavelets

Instructional Objectives

At the end of this lesson, the students should be able to:

1. Explain the space-frequency localization problem in sinusoidal transforms.
2. Explain the need for multi-resolution image analysis.
3. Define scaling functions.
4. Define functional subspace of scaling functions at a given scale.
5. Compute the scaled and translated versions of scaling functions.
6. Show the relationship between the functional subspaces of scaling functions at different scales.
7. Define wavelet functions.
8. Show the functional subspace relationship between scaling and wavelet functions.
9. Compute the scaled and translated versions of wavelet functions.
10. Express a continuous signal as a series expansion of scaling and wavelet basis functions.

10.0 Introduction

In lesson-9, we have studied the use of sinusoidal orthogonal transforms in energy compaction and consequently, image compression. In this family of transforms, Discrete Cosine Transforms (DCT) happens to be the most popular choice because of several advantages mentioned. However, as we noted, DCT has performance limitations in the form of blocking artifacts at very low bit rates. In recent years, a new transformation technique has emerged as popular alternatives to sinusoidal transforms at very low bit rates. Unlike DCT s and DFT s, which use sinusoidal waves as basis functions, this new variety of transformations use small waves of varying frequency and of limited extent, known as wavelets as basis. The wavelets can be scaled and shifted to analyze the spatial frequency contents of an image at different resolutions and positions. A wavelet can therefore perform analysis of an image at multiple resolutions, making it an effective tool in *multi-resolution analysis* of images. Furthermore, wavelet analysis performs what is known as *space-frequency localization* so that at any specified location in space, one can obtain its details in terms of frequency. It is like placing a magnifying glass above a photograph to explore the details around a specific location. The magnifying glass can be moved up or down to vary the extent of magnification, that is, the level of details and it can be slowly panned over the other locations of the photograph to extract those details. A classical sinusoidal transform does not allow such space-frequency localizations. If we consider the spatial array of pixels, it does not provide any spatial frequency information. On the other hand, the transformed array of coefficients contains spatial frequency information, but it does not give us any idea about the locations in the image where such spatial frequencies appear. The space-frequency localization capability of wavelets makes multi-resolution image analysis, representation and coding more efficient.

In this lesson, we shall develop the theory of scaling and wavelet functions which will form a basis of wavelet transforms, used as a tool for multi-resolution image analysis and coding, to be discussed in subsequent lessons. We are going to show the relationship between the scaling and the wavelet functions in terms of the functional spaces they span. Using this property, any continuous function of a variable can be expressed as a series summation of shifted and scaled versions of scaling and wavelet functions.

10.1 Need for multi-resolution image analysis

It is our common observation that the level of details within an image varies from location to location. Some locations contain significant details, where we require finer resolution for analysis and there are other locations, where a coarser resolution representation suffices. A multi-resolution representation of an image gives us a complete idea about the extent of the details existing at different locations from which we can choose our requirements of desired details. Multi-resolution representation facilitates efficient compression by exploiting the redundancies across the resolutions. Wavelet transforms is one of the popular, but not the only approach for multi-resolution image analysis. One can use any of the signal processing approaches to sub-band coding, such as Quadrature Mirror Filters (QMF) in multi-resolution analysis.

10.2 Scaling Functions and functional subspace

Any function *f*(*x*) can be analyzed as a linear combination of real-valued expansion functions $\varphi_k(x)$

$$f(x) = \sum_k \alpha_k \varphi_k(x) \quad \text{.......... (10.1)}$$

where *k* is an integer index of summation (finite or infinite), the α_k s are the real-valued expansion coefficients and $\{\varphi_k(x)\}$ forms an expansion set.

Let us compose a set of expansion functions $\{\varphi_{r,s}(x)\}$ through integer translations and binary scalings of the real, square-integrable function $\varphi(x)$, so that

$$\varphi_{r,s}(x) = 2^{r/2}\varphi(2^r x - s) \quad \text{.......... (10.2)}$$

where, $r, s \in Z$ (the integer space) and $\varphi(x) \in L^2(R)$ (the square-integrable real space). In the above equation, *s* controls the translation in integer steps and *r* controls the amplitude, as well as the width of the function in the *x*-direction. Increasing *r* by one decreases the width by one-half and increases the amplitude by $\sqrt{2}$. In other words, the index *r* scales the function and the set of functions $\{\varphi_{r,s}(x)\}$ obtained through equation (10.2) are referred to as *scaling functions*. By a wise choice of $\varphi(x)$, the set of functions $\{\varphi_{r,s}(x)\}$ can be made to cover the entire square-integrable real space $L^2(R)$. Hence, *if we choose any particular scale, say $r = r_0$, the set of functions $\{\varphi_{r_0,s}(x)\}$ obtained through integer translations can only cover a subspace of the entire $L^2(R)$. The subspace V_{r_0} so spanned is defined as the functional subspace of $\{\varphi_{r_0,s}(x)\}$ at a given scale* r_0. Since the width of the set of functions $\{\varphi_{r_0+1,s}(x)\}$ is half of that of the set of functions $\{\varphi_{r_0,s}(x)\}$, the latter can be analyzed by the former, but not the other way. Hence, the functional subspace spanned by $\{\varphi_{r_0+1,s}(x)\}$ contains the subspace $\{\varphi_{r_0,s}(x)\}$, that is, the subspace spanned by the scaling functions at lower scales is contained within the subspace spanned by those at higher scales and is given by the following nested relationship

$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\infty}$ …………………………………….. (10.3)

This subspace relationship is illustrated in Fig.10.1.

The expansion functions of subspace V_r can be expressed as a weighted summation of the functions of subspace V_{r+1} as follows

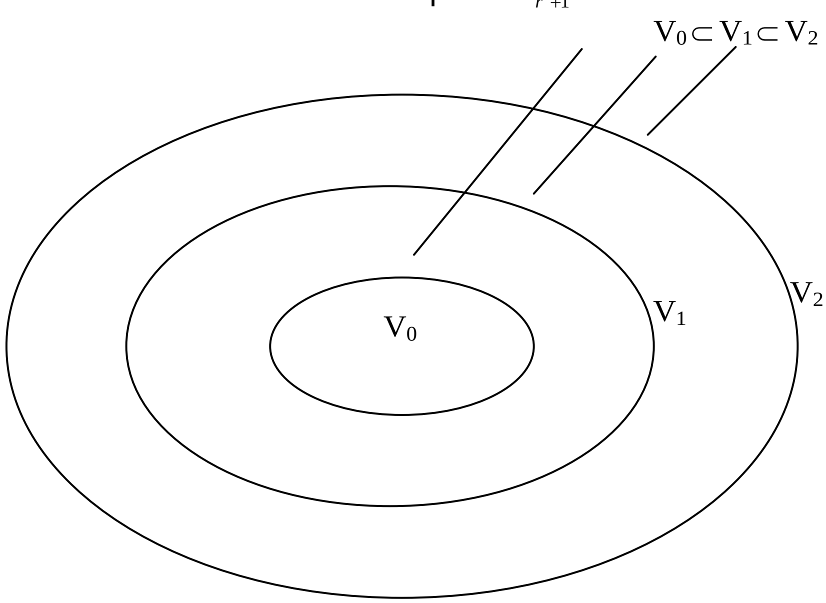


Fig 10.1 Subspace Relationship of Scaling Functions

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n) \quad \ldots\ldots (10.4)$$

where the $h_\varphi(n)$ are the wavelet function coefficients.

Let us consider the example of unit amplitude, unit width Haar scaling function, shown in Fig.10.2

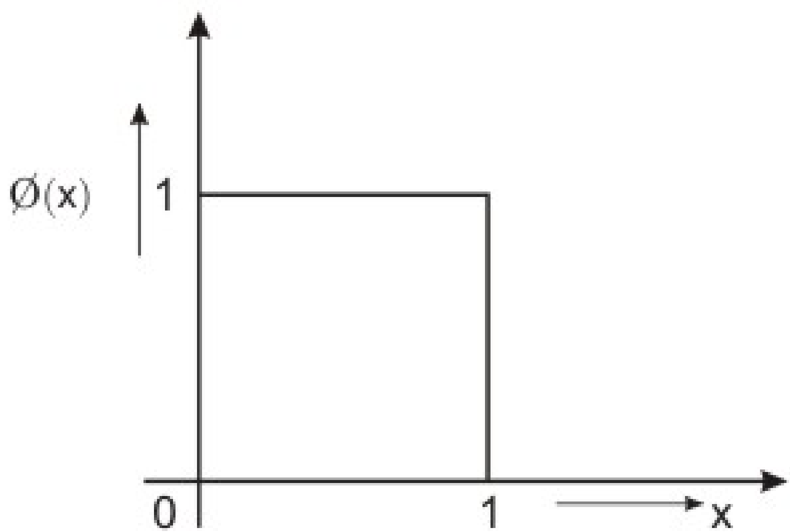


Fig 10.2 Haar Scaling Function

and mathematically defined as

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{otherwise} \end{cases} \quad \ldots\ldots (10.5)$$

By the mathematical definition of scaling functions, given in equation (10.1),

$$\varphi_{0,0}(x) = \varphi(x) \quad \ldots\ldots (10.6)$$

The functional subspace V_0 is spanned by the set $\{\varphi_{0,s}(x)\}$, each functional element of which represents a translated version of $\varphi_{0,0}(x)$ by an integer *s*. To obtain any scaled and translated version $\varphi_{r,s}(x)$ of the scaling function from $\varphi_{0,0}(x)$, it follows from the scaling function definition given in equation (10.2) that

(a) its amplitude should be $2^{r/2}$,

(b) its width should be $2^{-r/2}$,

(c) it should be positioned at $s.2^{-r/2}$

Fig.10.3, fig.10.4 and fig.10.5 show a few examples of these.

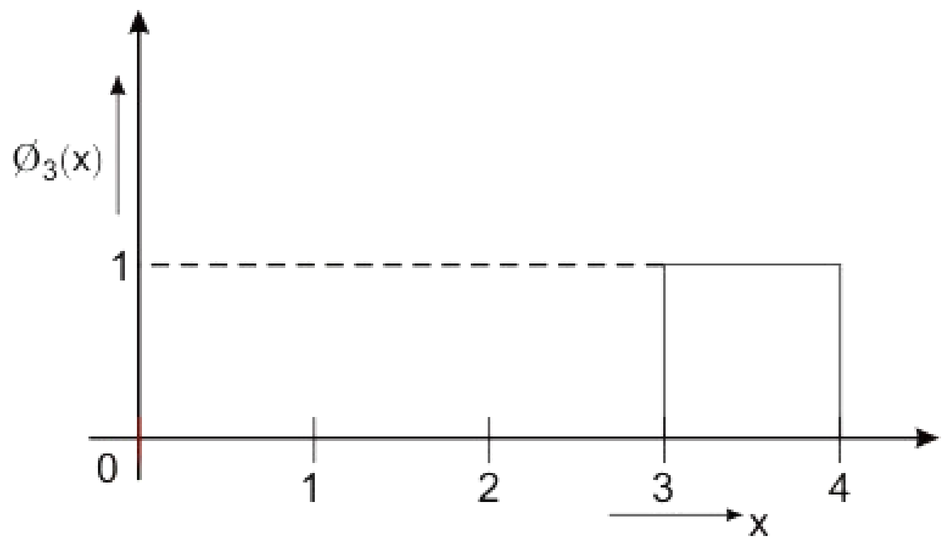


Fig 10.3: *Example of Scaling function [$\Phi_{0,3}(x)$]*

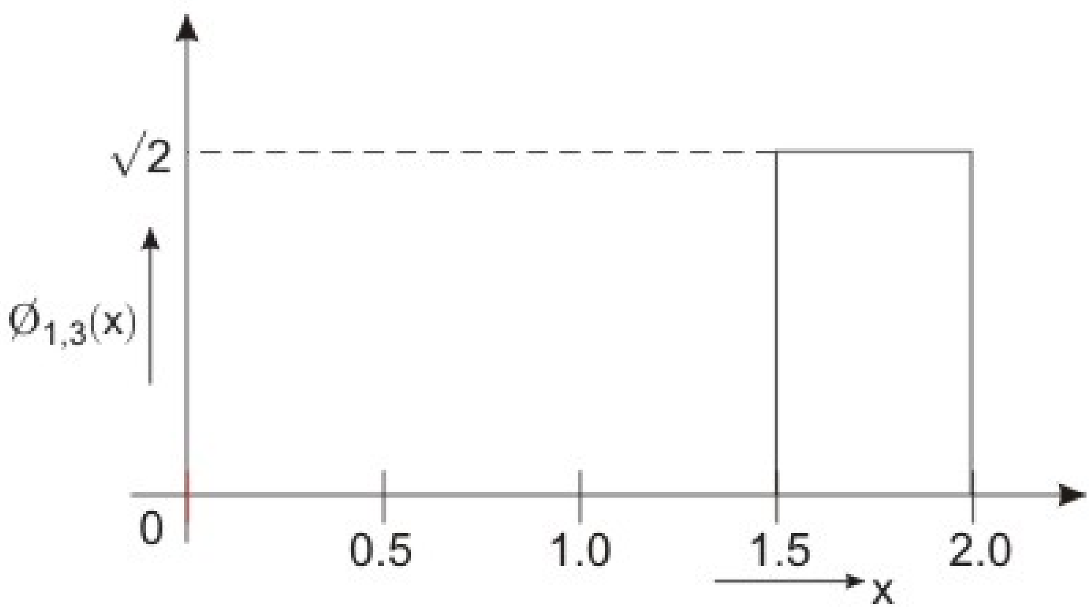


Fig 10.4: *Example of Scaling function [$\Phi_{1,3}(x)$]*

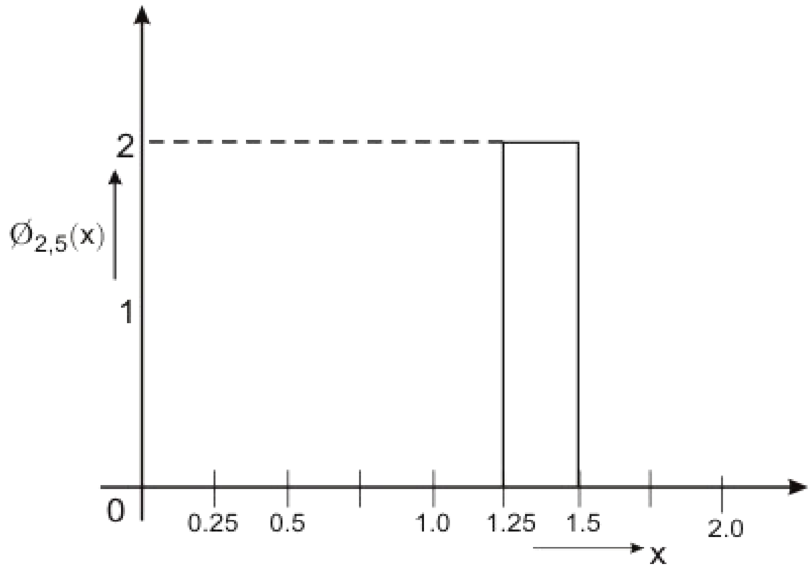


Fig 10.5: *Example of Scaling function [$\Phi_{2,5}(x)$]*

.3 Wavelet Functions:

A set of integer translated and binary scaled functions $\{\psi_{r,s}(x)\}$ *that spn the difference subspace between two adjacent scaling functions subspace is defined as a set of wavelet functions*. If we consider two adjacent subspaces V_r and V_{r+1}, the set of wavelets spanning the subspace W_r within these are given as

$$\psi_{r,s}(x) = 2^{r/2}\psi(2^r x - s) \qquad (10.7)$$

where, $s \in Z$ and $\psi(x) \in L^2(R)$. It may be noted that although the functional forms of equations (10.2) and (10.7) are the same, the scaling functions and the wavelet functions differ by their spanning subspaces. The relationship between scaling and wavelet function spaces is illustrated in fig.10.6

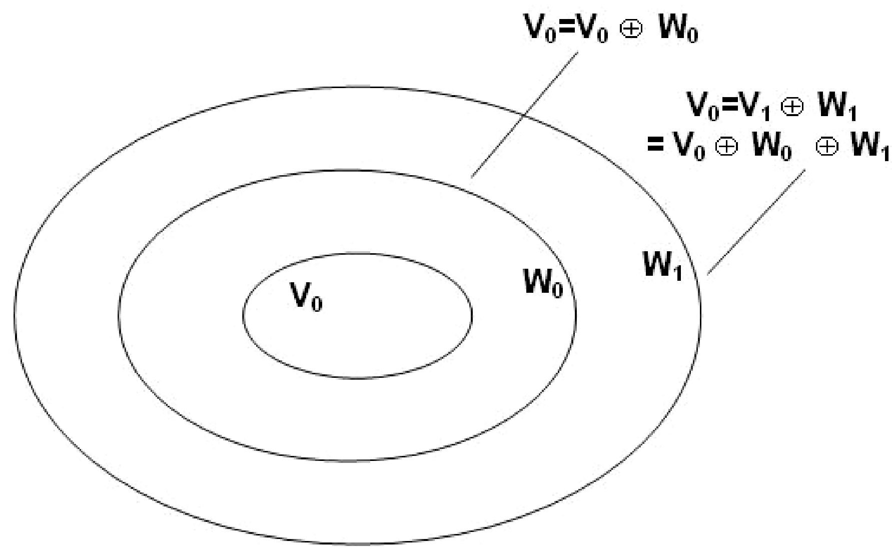


Fig 10.6: *Relationship between Scaling and wavelet Functions.*

and is given by

$V_{r+1} = V_r \oplus W_r$ …………………………………………………………………… (10.8)

where $\oplus$ indicates union of subspaces.

By recursively applying equation (10.7) to compute V_r, we can express the total measurable square-integrable space $L^2(R)$ as

$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots\cdots$ ……………………………………………………… (10.9)

Again, by repetitively applying equation (10.8) in (10.9), we can obtain alternative forms of expansion as

$$\begin{aligned} L^2(R) &= V_1 \oplus W_1 \oplus W_2 \oplus \cdots\cdots \\ &= V_2 \oplus W_2 \oplus W_3 \oplus \cdots\cdots \\ &= V_{r_0} \oplus W_{r_0} \oplus W_{r_0+1} \oplus \cdots\cdots \end{aligned}$$ …………………..………………...……………. (10.10)

Suppose that a function $f(x)$ to be analyzed belongs to the subspace V_1 but not V_0. In that case, the scaling functions of V_0 make a crude approximation of $f(x)$ and the wavelet functions of W_0 provide the details. In this sense, the scaling functions analyze $f(x)$ into its low-pass filtered form and the wavelet functions analyze $f(x)$ into its high-pass filtered form.

Since wavelet spaces reside within the spaces spanned by the next higher scaling functions, any wavelet function can be expressed as a weighted sum of shifted double-resolution scaling functions as follows

$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x-n)$ …………………………………………………..…. (10.11) where the $h_\psi(n)$ are the wavelet function coefficients. Using the conditions that the wavelets span the orthogonal complement spaces in fig.10.6 and the integer wavelet translates are orthogonal, it is possible to obtain relationship between $h_\varphi(n)$ and $h_\psi(n)$. Using the definition of Haar scaling function given in equation (10.5) and the solutions of $h_\varphi(n)$ and $h_\psi(n)$, the corresponding Haar wavelet function is obtained as

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{(10.12)}$$

Fig.10.7

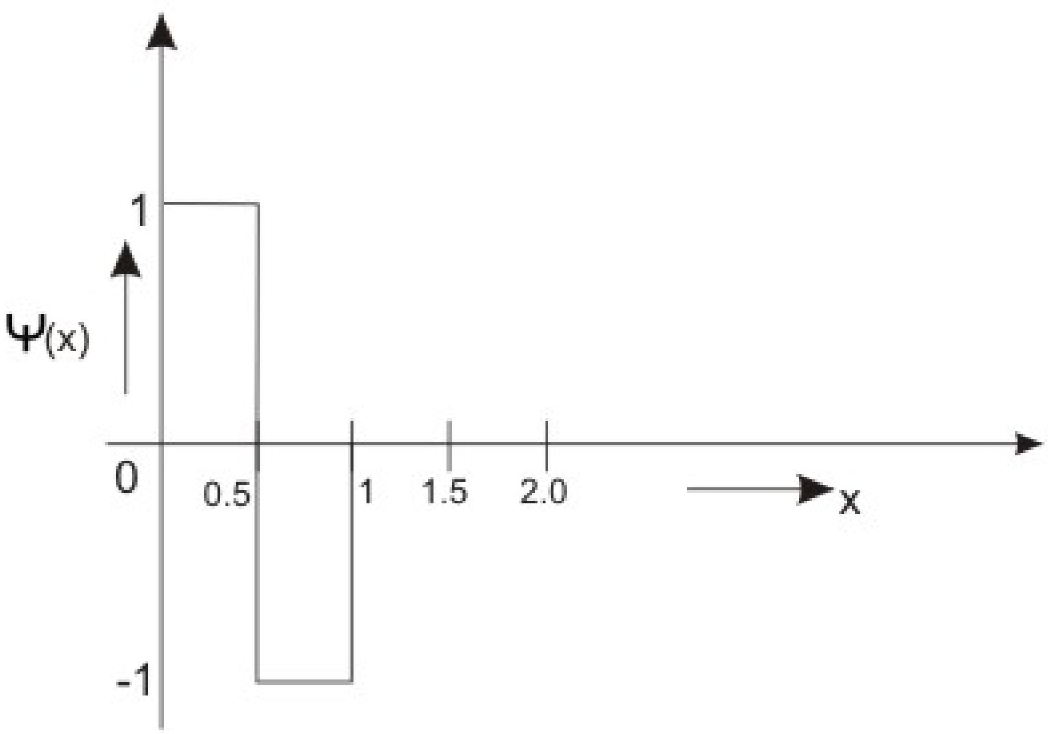


***Fig 10.7:** Haar Wavelet function*

shows equation (10.12) graphically. By the definition of equation (10.7),

$$\psi_{0,0}(x) = \psi(x) \quad \text{(10.13)}$$

Like the scaling functions, we can obtain binary scaled and integer shifted versions of wavelets by applying equation (10.7). Fig.10.8, fig.10.9 and fig.10.10 show few such examples.

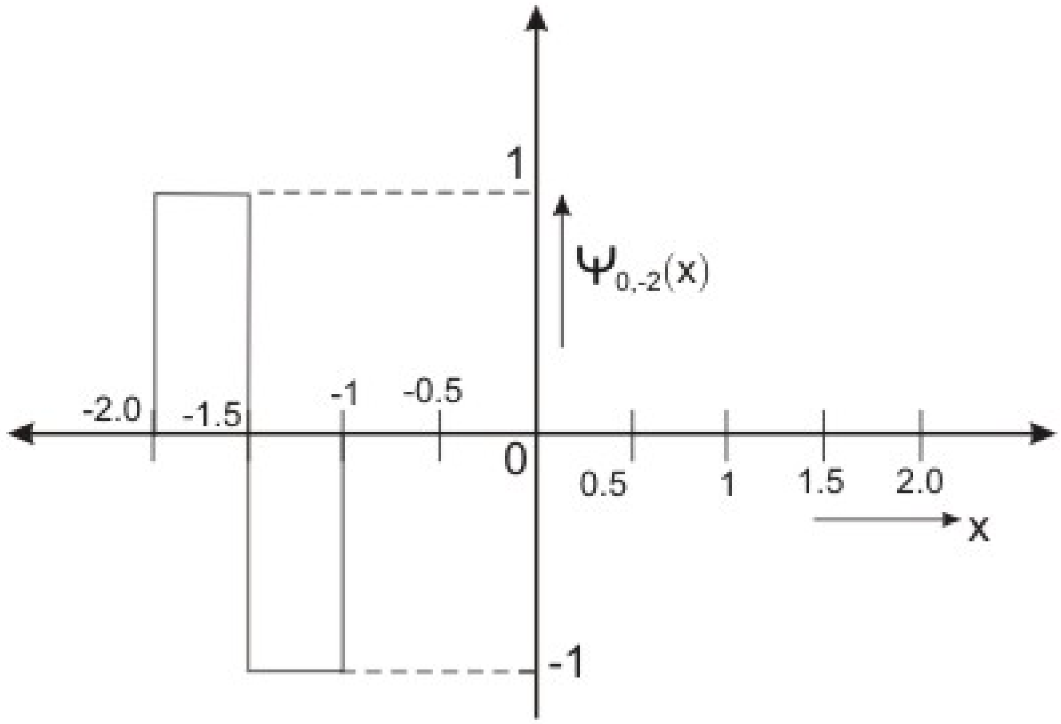


Fig 10.8: *Example of Haar Wavelet*

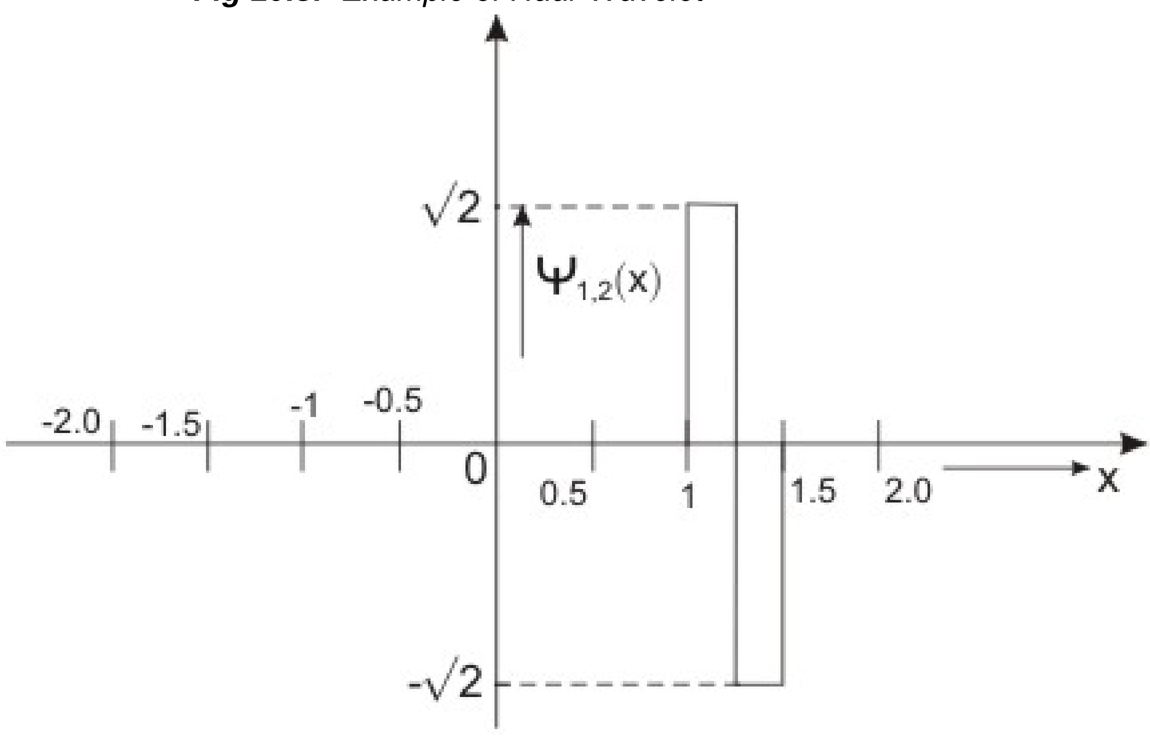


Fig 10.9: *Example of Haar Wavelet [$\Psi_{1,2}(x)$]*

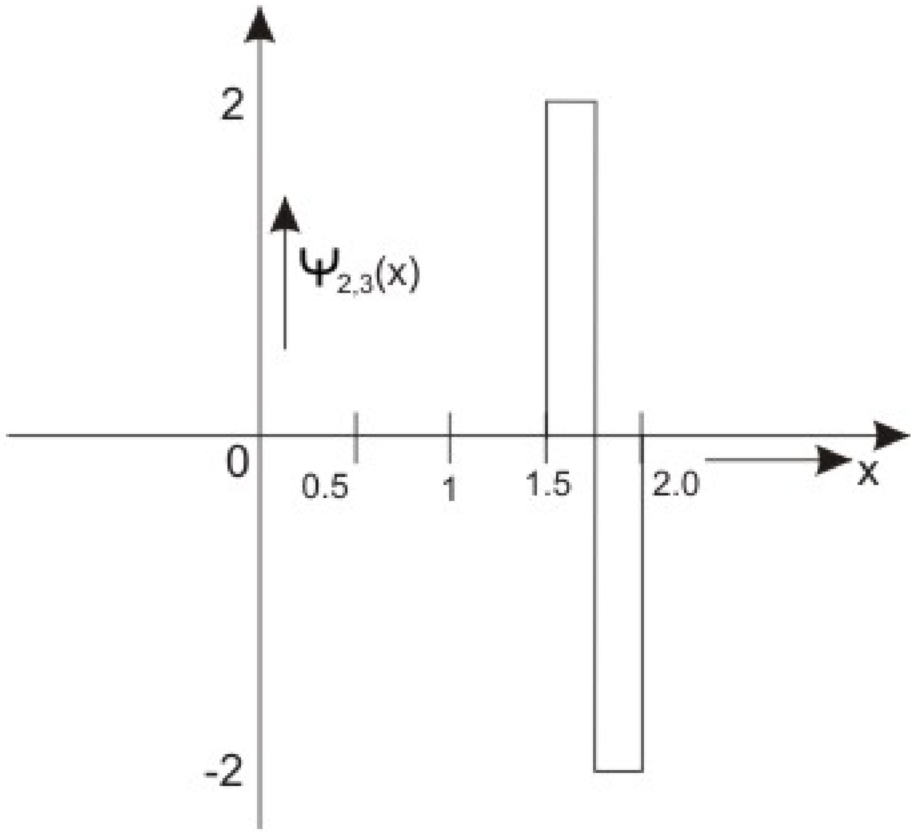


Fig 10.10: *Example of Haar Wavelet [$\Psi_{2,3}(x)$]*

10.4 <u>The wavelet series</u>:

In accordance with the functional subspace relationships shown in equation (10.10) and the definition of expansion functions in equation (10.1), any function $f(x) \in L^2(R)$ can be expressed as a series summation of scaling functions and wavelet functions as

$$f(x) = \sum_s a_{r_0,s}\varphi_{r_0,s}(x) + \sum_{r=r_0}^{\infty}\sum_s b_{r,s}\psi_{r,s}(x) \ldots\ldots\ldots\ldots \quad (10.14)$$

where, $a_{r_0,s}$ and $b_{r,s}$ are the corresponding expansion coefficients. In the above equation, the first term of the expansion involving the scaling functions provide approximations to *f*(*x*) at scale r_0 and the second term of expansion involving the wavelet functions add details to the approximation at r_0 and its higher scales. If the expansion functions form an orthonormal basis, which is often the case, the coefficients can be calculated as

$$a_{r_0,s} = \int f(x)\varphi_{r_0,s}(x)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

(10.15)

$$b_{r,s} = \int f(x)\psi_{r,s}(x)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

(10.16)

As an example, let us consider the wavelet series expansion of the following function:

$$f(x) = \begin{cases} e^x & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

(10.17)

using Haar scaling and wavelet functions.

By applying equations (10.15) and (10.16) on the function defined in (10.17), the expansion coefficients are obtained as follows:

$$a_{0,0} = \int_0^1 e^x \varphi_{0,0}(x) = \int_0^1 e^x dx = e^x \Big|_0^1 = e - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(10.18)

$$b_{0,0} = \int_0^1 e^x \psi_{0,0}(x) = \int_0^{0.5} e^x dx - \int_{0.5}^1 e^x dx = 2e^{0.5} - (e+1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(10.19)

$$b_{1,0} = \int_0^1 e^x \psi_{1,0}(x) = \int_0^{0.25} \sqrt{2}e^x dx - \int_{0.25}^{0.5} \sqrt{2}e^x dx = 2\sqrt{2}e^{0.25} - \sqrt{2}\left(e^{0.5} + 1\right) \ldots\ldots\ldots\ldots\ldots\ldots$$

(10.20)

$$b_{1,1} = \int_0^1 e^x \psi_{1,1}(x) = \int_{0.5}^{0.75} \sqrt{2}e^x dx - \int_{0.75}^{1} \sqrt{2}e^x dx = 2\sqrt{2}e^{0.75} - \sqrt{2}\left(e + e^{0.5}\right) \ldots\ldots\ldots\ldots\ldots..$$

(10.21)

Using the coefficients obtained in equations (10.18) to (10.21), the function *f*(*x*) can be realized as

$$f(x) = a_{0,0}\varphi_{0,0}(x) + b_{0,0}\psi_{0,0}(x) + b_{1,0}\psi_{1,0}(x) + b_{1,1}\psi_{1,1}(x) + \cdots\cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(10.22)

10.5 <u>Conclusion</u>:

In this lesson, we have presented the basic theory of the scaling and wavelet functions. It is shown that these functions can analyze a continuous valued, square-integrable signal in multiple resolutions. The scaling functions provide approximations or low-pass filtering of the signal and the wavelet functions add the details at multiple resolutions or perform high-pass filtering of the signal. Although the theory is presented for continuous, one-dimensional signals, it may be extended for discrete two-dimensional signals, which we require for multi-resolution image analysis and coding. The theory of subband decomposition for multi-resolution analysis will be presented in the next lesson. In the subsequent lessons, Discrete Wavelet Transforms (DWT) and its application in image coding and compression will be presented.

Questions

NOTE: The students are advised to thoroughly read this lesson first and then answer the following questions. Only after attempting all the questions, they should click to the solution button and verify their answers.

PART-A

A.1. Explain why sinusoidal transforms applied over images cannot have space-frequency localization.
A.2. Why is multi-resolution image analysis needed?
A.3. Define scaling functions.
A.4. Define functional subspace of scaling functions at a given scale.
A.5. Show the functional space relationship between scaling functions at different scales.
A.6. Define wavelet functions.
A.7. Show the functional space relationship between scaling and wavelet functions at different scales.
A.8. Express a continuous signal as a series expansion of scaling and wavelet basis functions.
A.9. Express the coefficients of wavelet series expansion in terms of the function and the orthonormal scaling and wavelet basis functions.

PART-B: Multiple Choice

In the following questions, click the best out of the four choices.
Radio buttons will be provided to the left of each choice. Only one out of the four buttons can be chosen.
B.1 Which of the following combination of scaling functions can be used to synthesize Haar scaling function $\varphi_{0,0}(x)$?

(A) $\varphi_{1,0}(x)+\varphi_{1,1}(x)$

(B) $\frac{1}{\sqrt{2}}\varphi_{1,0}(x)+\frac{1}{2}\varphi_{2,0}(x)+\frac{1}{2}\varphi_{2,1}(x)$

(C) $\frac{1}{\sqrt{2}}\varphi_{1,0}(x)+\frac{1}{2}\varphi_{2,2}(x)+\frac{1}{2}\varphi_{2,3}(x)$

(D) $\frac{1}{\sqrt{2}}\varphi_{1,0}(x)-\frac{1}{2}\varphi_{2,0}(x)-\frac{1}{2}\varphi_{2,1}(x)$

B.2 A square-integrable real-valued function $f(x)\in V_3$, but $f(x)\notin V_2$. Which of the following expansions can be used to realize *f*(*x*)?

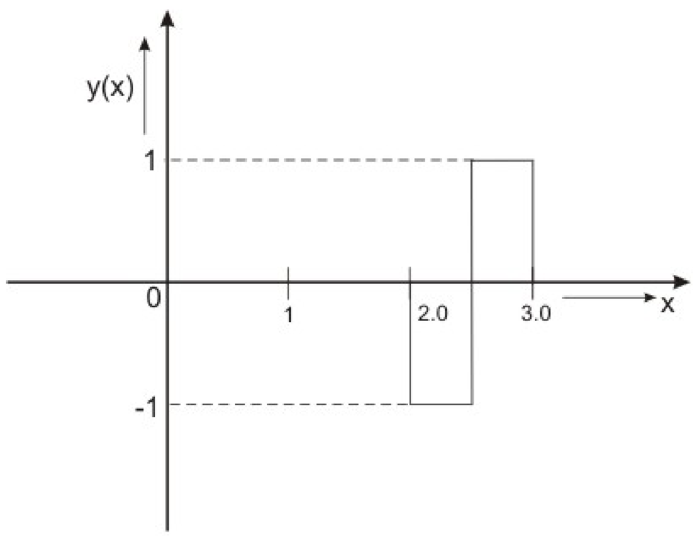
(A) $V_1 \oplus W_1 \oplus W_2$

(B) $V_0 \oplus W_1 \oplus W_2$

(C) $V_1 \oplus W_1 \oplus W_2 \oplus W_3$

(D) $V_0 \oplus W_1 \oplus W_2 \oplus W_3$

B.3 A function sketched below is to be represented by Haar wavelet function:



Its correct representation is:

(A) $\psi_{2,0}$

(B) $\psi_{0,2}$

(C) $\psi_{0,-2}$

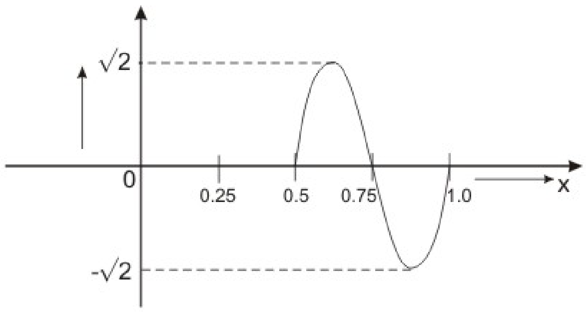
(D) $-\psi_{0,2}$

B.4 A wavelet function $\psi(x)$ is defined as follows:

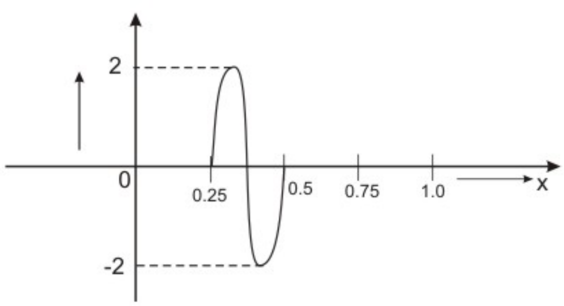
$$\psi(x)=\begin{cases}\sin 2\pi x & 0\le x<1\\ 0 & \text{otherwise}\end{cases}$$

Which one of the following graphs represents the function $\psi_{2,1}(x)$?

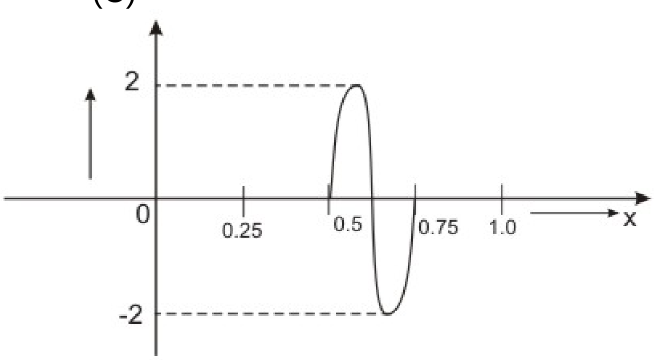
(A)



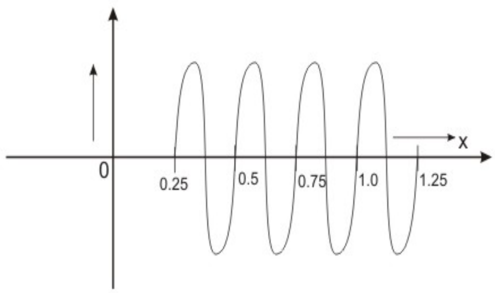
(B)



(C)



(D)



B.5 Which one of the statements is correct with reference to wavelet domain signal analysis:

(A) Scaling functions extract the details and wavelet functions perform approximation.

(B) Scaling functions perform approximations and wavelet functions extract the details.

(C) Wavelet functions perform approximations and also extraction of details.

(D) None of the above.

B.6 The coefficient of $\psi_{0,0}(x)$ in the series expansion of the following function

$$f(x)=\begin{cases} x^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is:

(A) 1/3

(B) 1/4

(C) -1/3

(D) -1/4

B.7 Which of the following functional space relationship is incorrect?

(A) $V_0 \subset W_0$

(B) $V_{-3} \subset V_0$

(C) $W_2 \subset V_3$

(D) $W_1 \subset V_\infty$

B.8 If the scale of a wavelet function is decreased by *r*, its width

(A) remains unchanged.

(B) gets multiplied by $2^{r/2}$.

(C) gets divided by $2^{r/2}$.

(D) gets multiplied by 2^r.

<u>PART-C:Problems</u>

C-1. A half sinusoid, defined by

$$f(x)=\begin{cases}\sin \pi x & 0 \le x < 1 \\ 0 & \text{otherwise}\end{cases}$$

is to be approximated by Haar scaling functions in function space V_0 and refined by Haar wavelet functions in space W_0 and W_1. Determine the scaling and wavelet functions and their associated coefficients.

C-2. Sketch the function realized by the following series of Haar scaling and wavelet functions:

$$f(x)=0.5\varphi_{0,0}(x)+0.3\psi_{1,1}(x)-0.3\psi_{1,2}(x)$$

SOLUTIONS

A.1
A.2
A.3
A.4
A.5
A.6

B.1 (C) B.2 (A) B.3 (D) B.4 (B)
B.5 (B) B.6 (D) B.7 (A) B.8 (B)

C.1
C.2

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