# Statistical Approach to Basis Function Truncation in Digital Interpolation Filters

F. Castillo, J. Arellano, and S. Sánchez

*Abstract*—In this paper an alternative analysis in the time domain is described and the results of the interpolation process are presented by means of functions that are based on the rule of conditional mathematical expectation and the covariance function. A comparison between the interpolation error caused by low order filters and the classic *sinc(t)* truncated function is also presented. When fewer samples are used, low-order filters have less error. If the number of samples increases, the *sinc(t)* type functions are a better alternative. Generally speaking there is an optimal filter for each input signal which depends on the filter length and covariance function of the signal. A novel scheme of work for adaptive interpolation filters is also presented.

*Keywords*—Interpolation, basis function, over-sampling.

## I. Introduction

THE main objective of digital to analog converters (DACs) is to reconstruct the signal accurately and efficiently. In practice, the implementation of reconstruction algorithms depends on the availability of the fastest processors so that the finished devices can be used in real-time applications.

In recent years, DACs have used techniques such as sigma delta modulation (SDM) and pulse width modulation (PWM), in which over-sampling and interpolation serve as an earlier phase of the conversion process [1]. This phase adds information (sampling points) between two adjacent samples, which results in an increase in the accuracy of the signal. Such precision is directly dependent on the efficiency of the interpolation algorithm.

The error between the reconstructed and the original signal must be as low as possible. One way to carry out the reconstruction process is to use low pass digital filters where a widespread option is the *sinc(t)* function as established by the classical sampling theorem.

A signal cannot be limited in both time and frequency domains simultaneously. The classic sampling theorem is limited in frequency; it uses the *sinc(t)* function which is continuous in the time domain. In the practical implementation of this theorem a truncated version of the *sinc(t)* function has been used, resulting in a frequency effect known as Gibbs' phenomenon [2]. The error caused by this effect can be significant in the reconstructed signal. On the other hand, the reduction of this error is directly associated with the filter order, which challenges the hardware in use.

This article presents an alternative analysis based on the rule of conditional mathematical expectation and the covariance function. As a result, a simpler design of basis functions is obtained. This also results in a lower error compared to the *sinc(t)* truncated function.

## II. Interpolation Process in Digital to Analog Converters

The purpose of the interpolation filter is to approach the signal that would have been obtained if it had been sampled at the over-sampling rate instead of at the Nyquist rate. The L interpolation order can vary from a few dozen to hundreds of points [3]; in any case, it is very important to use a robust reconstruction algorithm in order to get a low interpolation error as well as good computational efficiency.

A typical over-sampling DAC structure is shown in Fig. 1. The L interpolation order is a multiple of the Nyquist rate. Reducing the sampling rate by a factor M is known as decimation [4].



Fig. 1 Typical over-sampling DAC structure

Fig. 2 shows time and frequency signals through the interpolation process. Fig. 2 (a) shows the signal at the input of the digital interpolation filter. Fig. 2 (b) shows the signals at the filter output.



a)
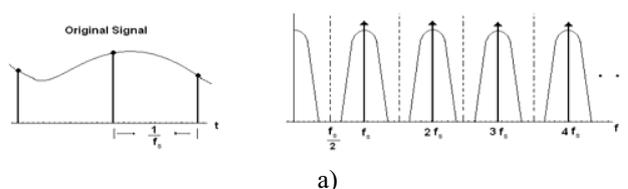
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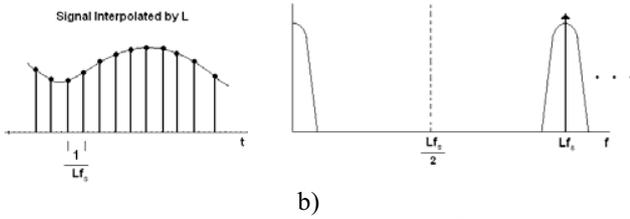
b)

Fig. 2 Interpolation effects. a) Original signal. b) Interpolated signal

One of the most widely used basis functions is the *sinc(t)* function, as described below.

## III. THE SINC($t$) FUNCTION

The classic sampling theorem is the fundamental basis of the sampling process and the signal's reconstruction [5]. It states that if *x(t)* is a low-pass signal limited to the band $(-W, W)$, the waveform is completely determined by the values taken at intervals of 1/2 *W*.

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \sin c\left[2W\left(t - \frac{n}{2W}\right)\right] \tag{1}$$

Where: $\sin c\left[2W\left(t - \frac{n}{2W}\right)\right] = B(t)$ is the basis reconstruction function.

Theoretically, the continuous function *x(t)* can be reconstructed from the samples using the interpolation function *sinc(t)*. This process can be described in terms of convolution as:

$$x(t) = Comb_{1/2W}\{x(t)\} * \sin c(2Wt) \tag{2}$$

There is a disadvantage to using this theorem in the interpolation process of practical systems: it requires the contribution of all the samples, going backwards and forwards to a given interpolation point. One possibility is to use a truncated version of the function and accept a *truncation error* in the output signal.

## IV. THE STATISTICAL METHOD

Given the values of a random signal *x(t)* which have been taken at regular intervals of time (truncated sequence), it is possible to observe some statistical characteristics such as the conditional expectation and the variance. These can be used to approximate a reconstruction function and an error function respectively [6].

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K_x(t, T_i) a_{ij}[x(T_j - m(T_j)] \tag{3}$$

$$\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K_x(t, T_i) a_{ij} K_x(T_j, t) \tag{4}$$

Where:

$$K_x(t_i, t_j) = \begin{bmatrix} K_x(t_1, t_1) & K_x(t_1, t_2) & ... & K_x(t_1, t_m) \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ K_x(t_m, t_1) & K_x(t_m, t_2) & ... & K_x(t_m, t_m) \end{bmatrix}$$

is the covariance matrix of the signal and

$$a = K_x^{-1}(t_i, t_j)$$

is the inverse matrix of $K_x$.

The Fourier transformation connects these results with their counterparts in the frequency domain.

$$K_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau} d\omega \tag{5a}$$

$$S_x(\omega) = \int_{-\infty}^{\infty} K_x(\tau) e^{-j\omega\tau} d\tau \tag{5b}$$

Where:
$S_x(\omega)$ is the power spectral density

This method is well known and has been used for sequence analysis of truncated Gaussian processes [7, 8]. Here, we focus on the basis functions analysis for interpolation filters and delimit the use of the *sinc(t)* function.

## V. IMPLEMENTATION

Comparing the expressions for the zero mean conditional expectation (3) and the sampling theorem (1) we have:

$$\hat{x}(t) = \tilde{m}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} K(t - T_i) a_{ij}[x(T_j)] = \sum_{j=1}^{N} x(T_j) B_j(t) \tag{6}$$

Where:

$$B_j(t) = \sum_{i=1}^{N} K(t - T_i) a_{ij}$$ is the filter impulse response in a linear system.

Since $B_j(t)$ depends on the covariance function, an analysis in the time domain is possible. A truncation of the signal will change the characteristics of the basis function $B_j(t)$. For the evaluation we use some known functions, such as the first, second and third order RC filters as well as the *sinc(t)* truncated function. These are used as covariance functions in order to calculate the covariance matrix $K_x$ and the $B_j(t)$ functions for filter 1, filter 2, filter 3 and the ideal filter (filter 4) respectively, see Fig. 3.

We use the mathematical expectation as an interpolation function; the variance is used as an error measure. The conditions considered for comparison are the unit covariance time and a normalized covariance function [9].

Due to covariance and inverse covariance matrices calculation, the computational cost of the statistical method was higher than the cost of the *sinc(t)* truncated function. The goal is to obtain the basis functions $B_j(t)$. As a matter of fact, once we have obtained them, they can be implemented in a linear system such as an interpolation filter in an embedded system [12]. In this case we do not need such matrices.

Fig. 4 shows a view of the developed program used to evaluate different basis functions using Matlab 7.0 language. It is possible to visualize the basis function, the interpolation function, and the interpolation error function.
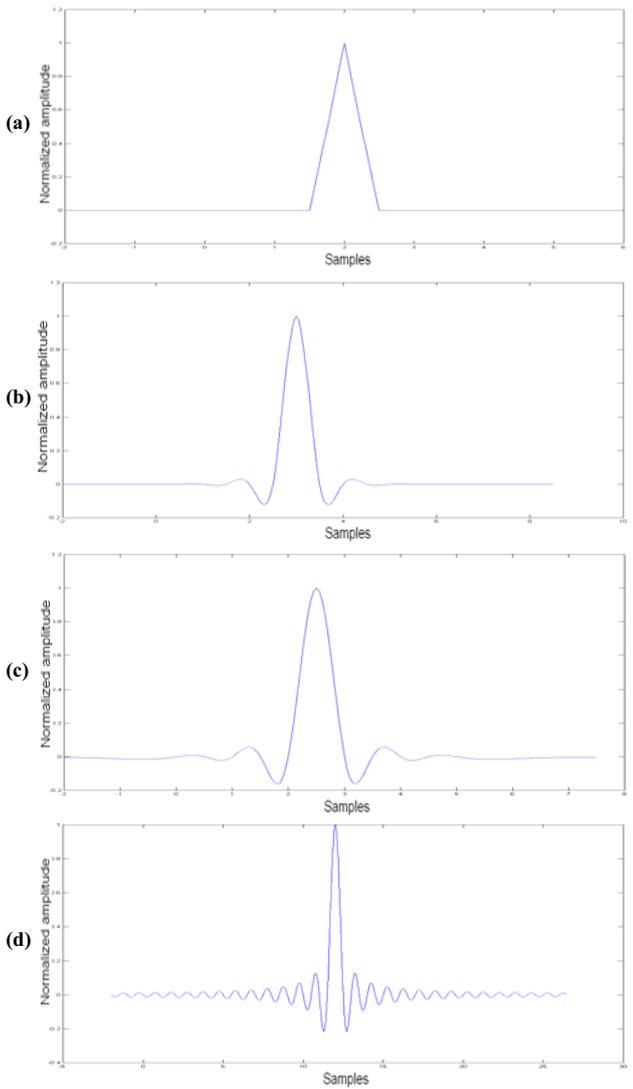


Fig. 3 Basis functions $B_j$, for different low pass filters. (a) Filter 1. (b) Filter 2. (c) Filter 3. (d) Ideal filter (truncated *sinc(t)*)

In order to compare the interpolation of the basis function proposed we used two different input functions. As a first case study we used a smooth function defined as:

$$f(n) = \cos(2\pi nT) + \frac{3}{2}\sin(\tfrac{8}{5}\pi nT) + 2\cos(\tfrac{6}{5}\pi nT) + \frac{5}{2}\sin(\tfrac{4}{5}\pi nT) + \frac{7}{2}\cos(\tfrac{2}{5}\pi nT)$$

$$(7)$$

This is a sine type function and it was sampled at the Nyquist rate. Table I shows the values obtained from (7).

TABLE I
VALUES OF F( $n$ ) FOR (7)

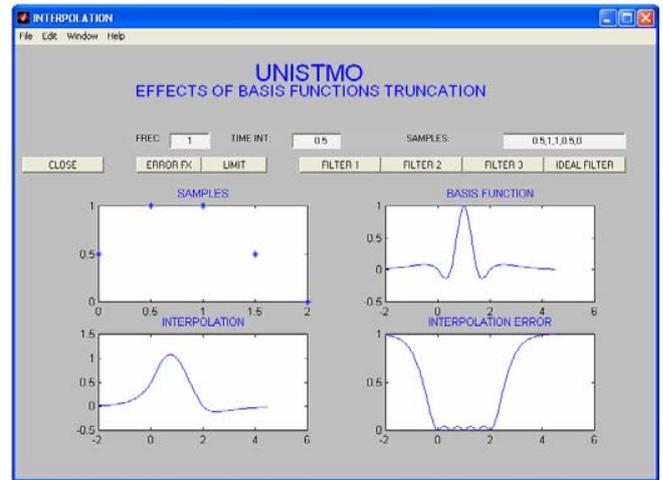| nT | f(n) |
|---|---|
| 0.0 | 6.5000 |
| 0.5 | 4.4728 |
| 1.5 | 0.5064 |
| 2.0 | -0.5064 |
| 2.5 | -4.4728 |
| 3.0 | -6.5000 |
| 3.5 | 2.0458 |
| 4.0 | -0.4206 |



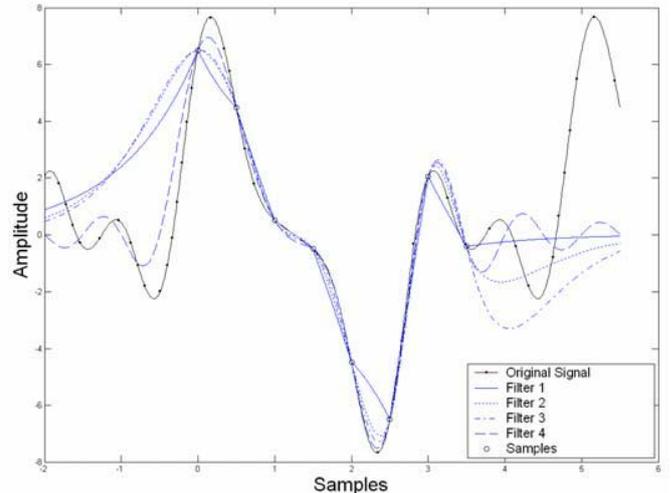Fig. 4  View of the developed program



Fig. 5 Interpolation process results for the first study case

Between samples 4 and 5, (central section) the *sinc(t)* function  is the nearest to the original signal; however, between samples 3 and 4, filter 1 is the best approximation. In some sections, some filters can be better than others.  (See Fig. 5).

As a second case study we used a section of the rectangular pulse function:

$$f(n) = 2G_\tau(nT) - 1 \qquad (8)$$

Table 2 shows the values obtained from (8).

TABLE II
VALUES OF F( $n$ ) FOR (8)

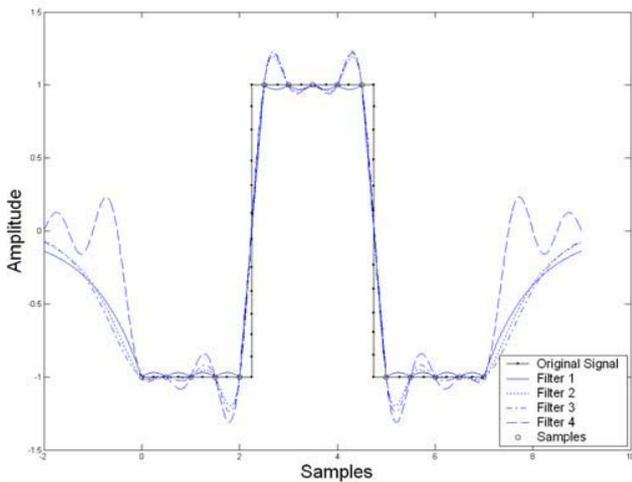| $nT$ | $f(n)$ |
|------|--------|
| 0.0 | -1 |
| 0.5 | -1 |
| 1.0 | -1 |
| 1.5 | -1 |
| 2.0 | -1 |
| 2.5 | 1 |
| 3.0 | 1 |
| 3.5 | 1 |
| 4.0 | 1 |
| 4.5 | 1 |
| 5.0 | -1 |
| 5.5 | -1 |
| 6.0 | -1 |
| 6.5 | -1 |
| 7.0 | -1 |



Fig. 6 Interpolation process results for the second study case

In this case, filter 1 is the nearest to the original signal, and the *sinc(t)* function is the worst case (see Fig. 6). It can be observed that the *sinc(t)* function had the most significant error at the beginning and end of the reconstructed signal, due to signal truncation.

High order filters are better when the reconstructed signal has a similar covariance to the basis function utilized. However, for chaotic signals, high order filters do not represent less error.

## VI. RESULT

Fig. 7 shows a comparison of the interpolation error for a limited quantity of samples. Minimal interpolation error is observed in the central section in each truncated sequence. Fig. 8 shows a comparison of interpolation error calculated in the central sample. In this case the sequence is unlimited but the filter length (basis function) is truncated. The amount of samples needed for reconstruction increases with the filter order (Fig. 3) in order to avoid the truncation error in a reconstructed signal. When this amount of samples is exceeded, the error remains independent of the number of samples. In the ideal filter the contribution of all samples is required.
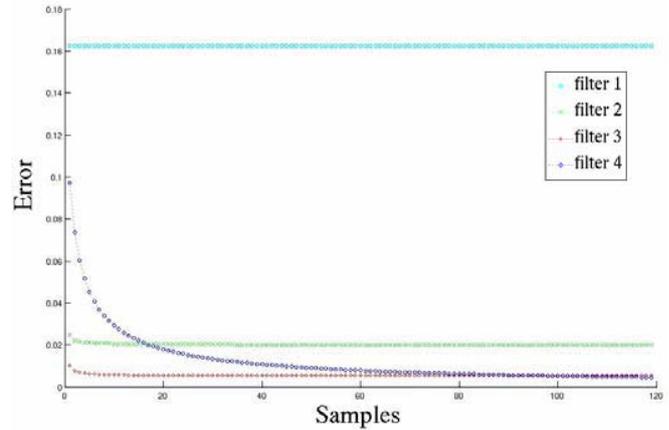


Fig. 7 Interpolation error for limited sequences

The basis functions $B_j$ are determined basically for the covariance matrices. Regardless of truncations, these functions are unique for each covariance function. As we increase the filter order, less error can be expected. When considering a small number of samples, the low-order filters showed less interpolation error than the *sinc(t)* function, since its basis function has not been truncated.

There is an optimal filter according to the covariance function of the input signal. In practice, filter design must take into account the interpolation order L and the digital filter length (taps) [11], but it might be difficult to have a basis function which matches the input signal.
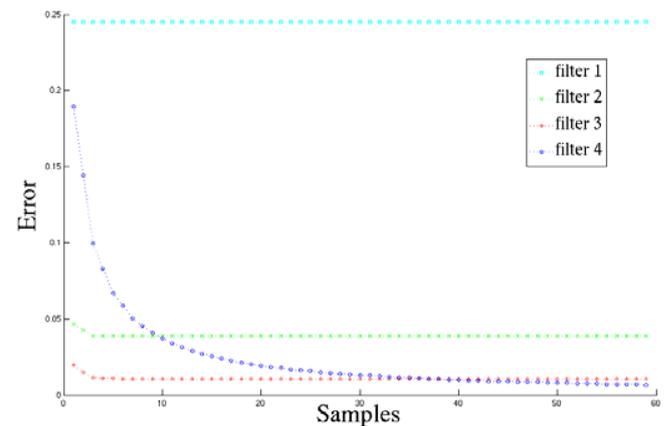


Fig. 8 Interpolation error for limited basis functions

Fig. 9 shows a proposal for an amended scheme, which considers the statistical characteristics of the input signal in order to calculate the coefficients of a dynamic basis function in an adaptive interpolation filter.
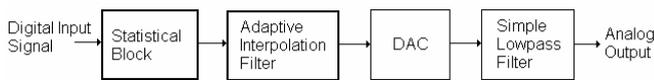
Fig. 9 Modified scheme for interpolation filter

Interpolation techniques in the time domain like Lagrange or splines are widely used. It is clear that in this case results can be enhanced by frequency analysis [11, 15]; however, when we use DSPs, efficient and fast algorithms are necessary for good performance. One possibility is to use covariance to define a time structure and to use it as design criteria. These results show that the basis functions have to match the input signal covariance in order to guarantee low interpolation error and minimal hardware requirements.

## VII. CONCLUSION

In this paper, statistical methods are used in order to study the effects of basis function truncation in the interpolation process. It is possible to implement interpolation functions based on the classic sampling theorem and the conditional mathematical expectation. It has been shown how this method provides simpler basis functions and causes less error in comparison to the classic *sinc(t)* truncated function. The use of the *sinc(t)* truncated function is limited to longer sequences. Statistical characteristics of the signal can be used in order to improve the interpolation filter's response, thus minimizing the truncation error. Generally speaking there is an optimal filter for each input signal which depends on the filter length and covariance function of the signal. These results can be used in designing digital interpolation filters in digital to analog converters.

## ACKNOWLEDGEMENT

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