

Module 4

Signal Representation and Baseband Processing

Lesson 15

Orthogonality

After reading this lesson, you will learn about

- *Basic concept of orthogonality and orthonormality;*
- *Strum - Lion;*
- *Slope overload distortion;*
- *Granular Noise;*
- *Condition for avoiding slope overloading;*

The Issue of Orthogonality

Let $f_m(x)$ and $f_n(x)$ be two real valued functions defined over the interval $a \leq x \leq b$. If the product $[f_m(x) \times f_n(x)]$ exists over the interval, the two functions are called orthogonal to each other in the interval $a \leq x \leq b$ when the following condition holds:

$$\int_b^a f_m(x) f_n(x) dx = 0, \quad m \neq n \quad 4.15.1$$

A set of real valued functions $f_1(x), f_2(x) \dots f_N(x)$ is called an orthogonal set over an interval $a \leq x \leq b$ if

- all the functions exist in that interval and
- all distinct pairs of the functions are orthogonal to each other over the interval, i.e.

$$\int_b^a f_i(x) f_j(x) dx = 0, \quad i = 1, 2, \dots; \quad j = 1, 2, \dots \text{ and } i \neq j \quad 4.15.2$$

The norm $\|f_m(x)\|$ of the function $f_m(x)$ is defined as,

$$\|f_m(x)\| = \sqrt{\int_b^a f_m^2(x) dx} \quad 4.15.3$$

An orthogonal set of functions $f_1(x), f_2(x) \dots f_N(x)$ is called an orthonormal set if,

$$\int_b^a f_m(x) \cdot f_n(x) = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \quad 4.15.4$$

An orthonormal set can be obtained from a corresponding orthogonal set of functions by dividing each function by its norm. Now, let us consider a set of real functions $f_1(x), f_2(x) \dots f_N(x)$ such that, for some non-negative weight function $w(x)$ over the interval $a \leq x \leq b$

$$\int_b^a f_m(x) \cdot f_n(x) \cdot w(x) dx = 0, \quad m \neq n \quad 4.15.5$$

Do f_i -s form an orthogonal set? We say that the f_i -s form an orthogonal set with respect to the weight function $w(x)$ over the interval $a \leq x \leq b$ by defining the norm as,

$$\|f_m(x)\| = \sqrt{\int_b^a f_m^2(x) \cdot w(x) dx} . \quad 4.15.6$$

The set of f_i -s is orthonormal with respect to $w(x)$ if the norm of each function is 1. The above extension of the idea of orthogonal set makes perfect sense. To see this, let

$$g_m(x) = \sqrt{w(x)} f_m(x), \text{ where } w(x) \text{ is a non-negative function.} \quad 4.15.7$$

It is now easy to verify that,

$$\int_b^a f_m(x) \cdot f_n(x) \cdot w(x) dx = \int_b^a g_m(x) \cdot g_n(x) dx = 0 . \quad 4.15.8$$

This implies that if we have orthogonal f_i -s over $a \leq x \leq b$, with respect to a non-negative weight function $w(x)$, then we can form an usual orthogonal set of f_i –s over the same interval $a \leq x \leq b$ by using the substitution,

$$g_m = \sqrt{w(x)} f_m(x)$$

Alternatively, an orthogonal set of g_i -s can be used to get an orthogonal set of f_i -s with respect to a specific non-negative weight function $w(x)$ over $a \leq x \leq b$ by the following substitution (provided $\sqrt{w(x)} \neq 0, a \leq x \leq b$):

$$f_m(x) = \frac{g_m(x)}{\sqrt{w(x)}} . \quad 4.15.9$$

A real orthogonal set can be generated by using the concepts of Sturm-Liouville (S-L) equation. The S-L problem is a boundary value problem in the form of a second order differential equation with boundary conditions. The differential equation is of the following form:

$$\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + [q(x) + \lambda \cdot \omega(x)] y = 0, \text{ for } a \leq x \leq b; \quad 4.15.10$$

It satisfies the following boundary conditions:

- i) $c_2 \frac{dy}{dx} + c_1 y = 0$; at $x = a$;
- ii) $d_2 \frac{dy}{dx} + d_1 y = 0$; at $x = b$;

Here c_1, c_2, d_1 and d_2 are real constants such that at least one of c_1 and c_2 is non zero and at least one of d_1 and d_2 is non zero.

The solution $y = 0$ is a trivial solution. All other solutions of the above equation subject to specific boundary conditions are known as characteristic functions or eigen-functions of the S-L problem. The values of the parameter ' λ ' for the non trivial solutions are known as characteristic values or eigen values. A very important property of the eigen-functions is that they are orthogonal.

Orthogonality Theorem:

Let the functions $p(x)$, $q(x)$ and $\omega(x)$ in the S-L equation (4.15.10) be real valued and continuous in the interval $a \leq x \leq b$. Let $y_m(x)$ and $y_n(x)$ be eigen functions of the S-L problem corresponding to distinct eigenvalues λ_m and λ_n respectively. Then, $y_m(x)$ and $y_n(x)$ are orthogonal over $a \leq x \leq b$ with respect to the weight function $w(x)$.

Further, if $p(x = a) = 0$, then the boundary condition (i) may be omitted and if $p(x = b) = 0$, then boundary condition (ii) may be omitted from the problem. If $p(x = a) = p(x = b)$, then the boundary condition can be simplified as,

$$y(a) = y(b) \text{ and } \left. \frac{dy}{dx} \right|_{x=a} = y'(a) = y'(b) = \left. \frac{dy}{dx} \right|_{x=b}$$

Another useful feature is that, the eigenvalues in the S-L problem, which in general may be complex based on the forms of $p(x)$, $q(x)$ and $w(x)$, are real valued when the weight function $\omega(x)$ is positive in the interval $a \leq x \leq b$ or always negative in the interval $a \leq x \leq b$

Examples of orthogonal sets:

Ex#1: We know that, for integer 'm' and 'n',

$$\int_{-\pi}^{\pi} \cos mx \cdot \cos nxdx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.1}$$

$$\int_{-\pi}^{\pi} \sin mx \cdot \sin nxdx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.2}$$

and $\int_{-\pi}^{\pi} \cos mx \cdot \sin nxdx = 0 \quad \text{E4.15.3}$

Let us consider equation E4.15.1 and rewrite it as:

$$\int_{-1/2f}^{1/2f} (\cos 2\pi mft) \cdot (\cos 2\pi nft) dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{E4.15.4}$$

by substituting $x = 2\pi ft = \omega t$ and $dx = 2\pi f dt = \omega dt$

Note that the functions 'cosmx' and 'cosnx' are orthogonal over the range 2π of the independent variable x and its integral multiple, i.e. $M \cdot 2\pi$, in general, where 'M' is an integer. This implies that equation (E4.15.4) is orthogonal in terms of the independent

variable 't' over the fundamental range $\frac{1}{f}$ and, in general, over $M \frac{1}{f} = M T_0$, where ' T_0 ' indicates the fundamental time interval over which $\cos 2\pi mft$ and $\cos 2\pi nft$ are orthogonal to each other. Now 'm' and 'n' can have a minimum difference '1' if

$$\int_{-T_0}^{T_0} (\cos 2\pi mft) \cdot (\cos 2\pi nft) dt = 0 \quad \text{E4.15.5}$$

i.e., $mf - nf = f = \frac{1}{T_0}$

So, if two cosine signals have a frequency difference 'f', then we may say,

$$\int_{-1/2f}^{1/2f} \cos 2\pi \left(f_c + \frac{f}{2}\right)t \cdot \cos 2\pi \left(f_c - \frac{f}{2}\right)t dt = 0 \quad \text{E4.15.6}$$

Re-writing equation (E4.15.6)

$$\int_{-T_0}^{T_0} \cos 2\pi \left(f_c + \frac{f}{2}\right)t \cdot \cos 2\pi \left(f_c - \frac{f}{2}\right)t dt = 0 \quad \text{where, } T_0 = \frac{1}{f}$$

Looking back at equation E4.15.5, we may write a general form for equation (E4.15.6):

$$\int_{-T_0/2}^{T_0/2} \cos 2\pi \left(f_c + p \frac{f}{2}\right)t \cdot \cos 2\pi \left(f_c - p \frac{f}{2}\right)t dt = 0 \quad \text{E 4.15.7}$$

where $mf = (n+p)f$ and 'p' is an integer.

Following similar observations on equation E4.15.2, one can say,

$$\int_{-T_0/2}^{T_0/2} \sin 2\pi \left(f_c + p \frac{f}{2}\right)t \cdot \sin 2\pi \left(f_c - p \frac{f}{2}\right)t dt = 0 \quad \text{E4.15.8}$$

Equation E4.15.3 may also be expressed as,

$$\begin{aligned} & \int_{-T_0/2}^{T_0/2} \cos 2\pi \left(f_c + p \frac{f}{2}\right)t \cdot \sin 2\pi \left(f_c - p \frac{f}{2}\right)t dt \\ &= \int_{-T_0/2}^{T_0/2} \sin 2\pi \left(f_c + p \frac{f}{2}\right)t \cdot \cos 2\pi \left(f_c - p \frac{f}{2}\right)t dt = 0 \end{aligned} \quad \text{E4.15.9}$$

Let us define $s_1 = \cos 2\pi \left(f_c + p \frac{f}{2}\right)t$, $s_2 = \cos 2\pi \left(f_c - p \frac{f}{2}\right)t$, $s_3 = \sin 2\pi \left(f_c + p \frac{f}{2}\right)t$

and $s_4 = \sin 2\pi \left(f_c - p \frac{f}{2}\right)t$. Can we use the above observations on orthogonality to

distinguish among 's_i-s' over a decision interval of $T_5 = T_0 = \frac{1}{f}$?

Ex#2: $x_1(t) = 1.0$ for $0 \leq t \leq T/2$ and zero elsewhere, ■
 $x_2(t) = 1.0$ for $T/2 \leq t \leq T$ and zero elsewhere, ■

Ex#3: $x_1(t) = 1.0$ for $0 \leq t \leq T/2$ and $x_1(t) = -1.0$ for $T/2 < t \leq T$, while
 $x_2(t) = -1.0$ for $0 \leq t \leq T$ ■

Importance of the concepts of Orthogonality in Digital Communications

- In the design and selection of information bearing pulses, orthogonality over a symbol duration may be used to advantage for deriving efficient symbol-by-symbol demodulation scheme.
- Performance analysis of several modulation demodulation schemes can be carried out if the information-bearing signal waveforms are known to be orthogonal to each other.
- The concepts of orthogonality can be used to advantage in the design and selection of single and multiple carriers for modulation, transmission and reception.

Orthogonality in a complex domain

Let, $z_1(t) = x_1(t) + jy_1(t)$ and $z_2(t) = x_2(t) + jy_2(t)$

Now, $x_1(t) = \frac{z_1(t) + z_1^*(t)}{2}$ and $x_2(t) = \frac{z_2(t) + z_2^*(t)}{2}$

If x_1 and x_2 are orthogonal to each other over $a \leq t \leq b$,

$$\int_a^b x_1(t) \cdot x_2(t) dt = 0$$

$$\text{i.e., } \int_a^b [z_1(t) + z_1^*(t)][z_2(t) + z_2^*(t)] dt = 0$$

$$\text{or, } \int_a^b [z_1(t) \cdot z_2(t) + z_1(t) \cdot z_2^*(t) + z_1^*(t) \cdot z_2(t) + z_1^*(t) \cdot z_2^*(t)] dt = 0$$

Let us consider a complex function

$$\begin{aligned} z_1(t) &= x(t) + jy(t), \quad a \leq t \leq b \\ &= r(t)[\cos \Phi(t) + j \sin \Phi(t)] \end{aligned}$$

where, $r(t) = |\tilde{z}(t)|$, a non – negative function of ‘t’.

$$\therefore x(t) = r(t) \cos \Phi(t) \quad \text{and} \quad y(t) = r(t) \sin \Phi(t)$$

$$\text{Now, } \int_a^b x(t) \cdot y(t) dt = \int_a^b r^2(t) \cdot \cos \Phi(t) \cdot \sin \Phi(t) dt$$

We know that $\cos \theta$ & $\sin \theta$ are orthogonal to each other over $-\pi \leq \theta < \pi$, i.e.,

$$\int_{-\pi}^{\pi} \cos \theta \cdot \sin \theta d\theta = 0$$

So, using a constant weight function $w = r$, which is non-negative, we may say

$$\int_{-\pi}^{\pi} r^2 \cos \theta \cdot \sin \theta d\theta = 0$$

Now, $x = r \cos \theta$ and $y = r \sin \theta$ are also orthogonal over $-\pi \leq \theta < \pi$.

Now, let 'θ' be a continuous function of 't' over $-\pi \leq \theta < \pi$. And,

$$\theta|_{t=a} = \theta_a = -\pi \quad \text{and} \quad \theta|_{t=b} = \theta_b = \pi$$

Assuming a linear relationship, let, $\theta(t) = 2\pi ft$

$$\therefore d\theta(t) = 2\pi f dt$$

Under these conditions, we see,

$$\begin{aligned} \int_a^b r^2(t) \cdot \cos \Phi(t) \cdot \sin \Phi(t) dt &= \frac{1}{2\pi f} \int_{-\pi}^{\pi} r^2(t) \cdot \cos \Phi(t) \cdot \sin \Phi(t) d\Phi \\ &= \frac{1}{2\pi f} \int_{-\pi}^{\pi} r^2 \cdot \cos \Phi(t) \cdot \sin \Phi(t) d\Phi = 0 \end{aligned}$$

i.e., $x(t)$ and $y(t)$ are orthogonal over the interval $-\frac{1}{2f} \leq t \leq \frac{1}{2f}$ or $\frac{T}{2} \leq t \leq \frac{T}{2}$

So, if $\tilde{z}(t) = x(t) + jy(t)$ represents a phasor in the complex plane rotating at a uniform frequency of 'f', then $x(t)$ and $y(t)$ are orthogonal to each other over the interval

$$-\frac{1}{2f} \leq t \leq \frac{1}{2f} \quad \text{or, equivalently} \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \quad \text{where} \quad T = \frac{1}{f} \quad \text{i.e.,} \quad \int_{-T/2}^{T/2} x(t) \cdot y(t) dt = 0$$

Now, let us consider two complex functions:

$$\tilde{z}_1(t) = x_1(t) + jy_1(t) = |\tilde{z}_1(t)| e^{j\Phi_1(t)}$$

$$\text{and} \quad \tilde{z}_2(t) = x_2(t) + jy_2(t) = |\tilde{z}_2(t)| e^{j\Phi_2(t)}$$

$[x_1(t), y_1(t)]$ and $[x_2(t), y_2(t)]$ are orthogonal pairs over the interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$. So, $\tilde{z}_1(t)$

and $\tilde{z}_2(t)$ may be viewed as two phasors rotating with equal speed.

Now, two static phasors are orthogonal to each other if their dot or scalar product is zero, i.e.,

$$\overline{A \cdot B} = |A||B| \cos \gamma = A_x \cdot B_x + A_y \cdot B_y = 0, \quad \text{where '}\gamma\text{' is the angle between } \overline{A} \text{ and } \overline{B}$$

In general, two complex functions $\tilde{z}_1(t)$ and $\tilde{z}_2(t)$ with finite energy are said to be orthogonal to each other over an interval $a \leq t \leq b$, if

$$\int_a^b \tilde{z}_1(t) \cdot \tilde{z}_2^*(t) dt = 0$$

Problems

Q4.15.1) Verify whether two signals are orthogonal over one time period of the signal with smallest frequency signal.

i) $X_1(t) = \cos 2\pi ft$ and $X_2(t) = \sin 2\pi ft$

ii) $X_1(t) = \cos 2\pi ft$ and $X_2(t) = \cos (2\pi ft + \frac{\pi}{3})$

iii) $X_1(t) = \cos 2\pi ft$ and $X_2(t) = \cos (4\pi ft + \frac{\pi}{4})$

iv) $X_1(t) = \sin 4\pi ft$ and $X_2(t) = -\cos (\pi ft - \frac{\pi}{6})$

Source: <http://nptel.ac.in/courses/Webcourse-contents/IIT%20Kharagpur/Digi%20Comm/pdf-m-4/m4115.pdf>