Module 4

Signal Representation and Baseband Processing

Lesson
19
Maximum Likelihood Detection and Correlation Receiver

After reading this lesson, you will learn about

- ***Principle of Maximum Likelihood (ML) detection;***
- ***Likelihood function;***
- ***Correlation receiver;***
- ***Vector receiver;***

Maximum likelihood (ML) detection:

We start with the following assumptions:

- Number of information-bearing signals (symbols), designed after the G-S-O approach, is 'M' and one of these 'M' signals is received from the AWGN channel in each time slot of 'T'-sec. Let the messages be denoted by m_i, i = 1, 2, …, M. Each message, as discussed in an earlier lesson, may be represented by a group of bits, e.g. by a group of 'm' bits each such that $2^m = M$.

- All symbols are equi- probable. This is not a grave assumption since, if the input message probabilities are different and known, they can be incorporated following Bayesian approach. However, for a bandwidth efficient transmission scheme, as is often needed in wireless systems, the source coding operation should be emphasized to ensure that all the symbols are independent and equally likely. Alternatively, the number of symbols, M, may also be decided appropriately to approach this desirable condition.

- AWGN process with a mean = 0 and double-sided psd No/2. Let w(t) denote a noise sample function over $0 \le t < T$.

Let 'R(t)' denote the received random process with sample function over a symbol duration denoted as r(t), $0 \le t \le T$. Now, a received sample function can be expressed in terms of the corresponding transmitted information-bearing symbol, say $s_i(t)$, and a sample function w(t) of the Gaussian noise process simply as:

$$r(t) = s_i(t) + w(t), \quad 0 \le t < T \qquad 4.19.1$$

At the receiver, we do not know which $s_i(t)$ has been transmitted over the interval $0 \le t < T$. So, the job of an efficient receiver is to make 'best estimate' of transmitted signal $[s_i(t)]$ upon receiving r(t) and to repeat the same process during all successive symbol intervals. This problem can be explained nicely using the concept of '*signal space*', introduced earlier in Lesson #16. Depending on the modulation and transmission strategy, the receiver usually has the knowledge about the signal constellation that is in use. This also means that the receiver knows all the nominal basis functions used by the transmitter. For convenience, we will mostly consider a transmission strategy involving two basis functions, $\boldsymbol{\varphi_1}$ and $\boldsymbol{\varphi_2}$ (described now as unit vectors) for explanation though most of the discussion will hold for any number of basis functions. **Fig.4.19.1** shows a two-dimensional signal space showing a signal vector $\bar{s_i}$ and a received vector $\bar{r}$. Note the noise vector $\bar{\omega}$. as well.

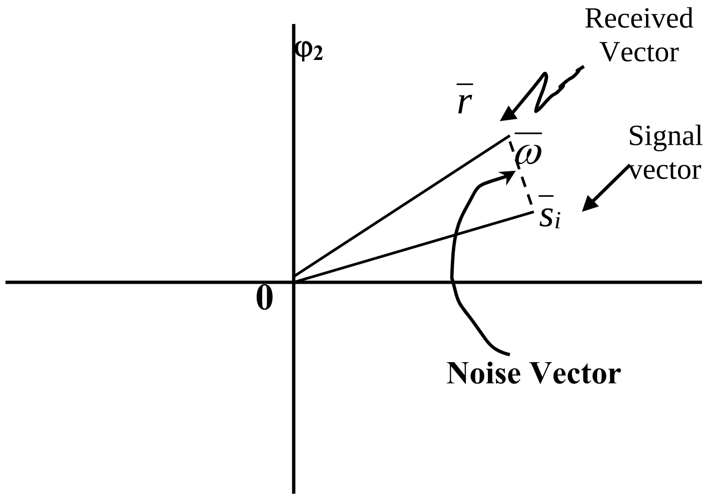

Fig. 4. 19.1 *Signal space showing a signal vector* $\bar{s}_i$ *and a received vector* $\bar{r}$

The job of the receiver can now be formally restated as: Given received signal vectors $\bar{r}$, find estimates $\hat{m}_i$ for all valid transmit symbols 'm_i-s' once in each symbol duration in a way that would minimize the probability of erroneous decision of a symbol on an average (continuous transmission of symbols is implicit).

The principle of Maximum Likelihood (ML) detection provides a general solution to this problem and leads naturally to the structure of an optimum receiver. When the receiver takes a decision that $\hat{m} = m_i$, the associated probability of symbol decision error may be expressed as: $Pe\left(m_i, \bar{r}\right)$ = probability of decision on receiving $\bar{r}$ that 'm_i' was transmitted = Pr (m_i not sent $|\bar{r}$) = 1 – Pr (m_i sent $|\bar{r}$).

In the above, Pr (m_i not sent $|\bar{r}$) denotes the probability that 'm_i' was not transmitted while $\bar{r}$ is received. So, an optimum decision rule may heuristically be framed as:

Set $\hat{m} = m_i$ if Pr (m_i sent $|\bar{r}$) $\geq$ Pr (m_k sent $|\bar{r}$), for all $k \neq i$ 4.19.2

This decision rule is known as *maximum a posteriori probability* rule. This rule requires the receiver to determine the probability of transmission of a message from the received vector. Now, for practical convenience, we invoke Bayes' rule to obtain an equivalent statement of optimum decision rule in terms of a priori probability:

$$\underbrace{\Pr\left(m_i|\bar{r}\right)}_{\substack{\text{a posteriori prob.} \\ \text{of 'm}_i\text{' given } \bar{r}}} \; \underbrace{\Pr\left(\bar{r}\right)}_{\substack{\text{joint probability} \\ \text{of } \bar{r}}} = \underbrace{\Pr\left(\bar{r}|m_i\right)}_{\substack{\text{a priori prob.} \\ \text{of } \bar{r} \text{ given 'm}_i\text{'}}} \underbrace{\Pr\left(m_i\right)}_{\frac{1}{M}} \qquad 4.19.3$$

$\Pr\left(m_i|\bar{r}\right)$: A posteriori probability of m_i given $\vec{r}$

$Pr(\bar{r})$: Joint pdf of $\vec{r}$, defined over the entire set of signals { $s_i(t)$ }; independent of any specific message 'm_i '

$Pr(\bar{r}|m_i)$: Probability that a specific $\bar{r}$ will be received if the message m_i is transmitted; known as the a priori probability of $\vec{r}$ given m_i

$Pr(m_i)$: $1/M$

From Eq. 4.19.3, we see that determination of maximum a posteriori probability is equivalent to determination of maximum a priori probability $Pr(\bar{r}|m_i)$. This a priori probability is also known as the '*likelihood function*'.

So the decision rule can equivalently be stated as:

Set $\hat{m} = m_i$ if $Pr(\bar{r}|m_i)$ is maximum for k = i

Usually, $\ln\left[p_r\left(\vec{r}|m_k\right)\right]$, i.e. natural logarithm of the likelihood function is considered. As the likelihood function is non-negative, another equivalent form for the decision rule is:

$$\text{Set } \hat{m} = m_i \text{ if } \ln[Pr(\bar{r}|m_i)] \text{ is maximum for k = i} \qquad 4.19.4$$

A '*Maximum Likelihood Detector*' realizes the above decision rule.Towards this, the signal space is divides in M decision regions, Z_i , i = 1, 2, …, M such that,

$$\begin{cases} \text{vector } \bar{r} \text{ lies inside } 'Z_i' \text{ if,} \\ \ln\left[P_r\left(\bar{r}|m_k\right)\right] \text{is maximum for k = i} \end{cases} \qquad 4.19.5$$

Fig. 4.19.2 indicates two decision zones in a two-dimensional signal space. The received vector $\bar{r}$ lies inside region Z_i if $\ln\left[p_r\left(\vec{r}|m_k\right)\right]$ is maximum for k = i.

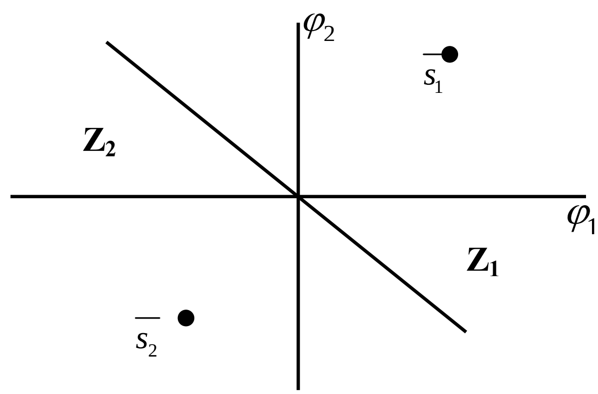

Fig. 4.19.2 *Decision zones in a two-dimensional signal space*

Now for an AWGN channel, the following statement is equivalent to ML decision:

Received vector $\vec{r}$ lies inside decision region Z_i

if, $\sum_{j=1}^{N}\left(r_j - s_{kj}\right)^2$ is minimum for k = i 4.19.6

That is, the decision rule simply is to choose the signal point $\bar{s}_i$ if the received vector $\vec{r}$ is closest to $\bar{s}_i$ in terms of Euclidean distance. So, it appears that Euclidean distances of a received vector $\vec{r}$ from all the signal points are to be determined for optimum decision-making. This can, however, be simplified. Note that, on expansion we get,

$$\sum_{j=1}^{N}\left(r_j - s_{kj}\right)^2 = \sum_{j=1}^{N} r_j^2 - 2\sum_{j=1}^{N} r_j . s_{kj} + \sum_{j=1}^{N} s^2_{kj} \qquad 4.19.7$$

It is interesting that, the first term on the R.H.S, i.e., $\sum_{j=1}^{N} r_j^2$ is independent of 'k' and hence need not be computed for our purpose. The second term, $2\sum_{j=1}^{N} r_j . s_{kj}$ is the inner product of two vectors. The third term, i.e. $\sum_{j=1}^{N} s_{kj}^2$ is the energy of the k-th symbol. If the modulation format is so chosen that all symbols carry same energy, this term also need not be computed. We will see in Module #5 that many popular digital modulation schemes such as BPSK, QPSK exhibit this property in a linear time invariant channel.

So, a convenient observation is: the received vector $\vec{r}$ lies in decision region Z_i if, $\left(\sum_{j=1}^{N} r_j s_{kj} - \frac{1}{2}E_k\right)$ is maximum for k = i

That is, a convenient form of the ML decision rule is:

$$\text{Choose } \hat{m} = m_i \text{ if } \left(\sum_{j=1}^{N} r_j s_{kj} - \frac{1}{2}E_k\right) \text{ is maximum for } k = i \qquad 4.19.8$$

A *Correlation Receiver*, consisting of a Correlation Detector and a Vector Receiver implements the M – L decision rule [4.19.8] by, (a) first finding $\vec{r}$ with a correlation detector and then (b) computing the metric in [4.19.8] and taking decision in a vector receiver. **Fig. 4.19.3** shows the structure of a Correlation Detector for determining the received vector $\bar{r}$ from the received signal r(t). **Fig. 4.19.4** highlights the operation of a Vector Receiver.

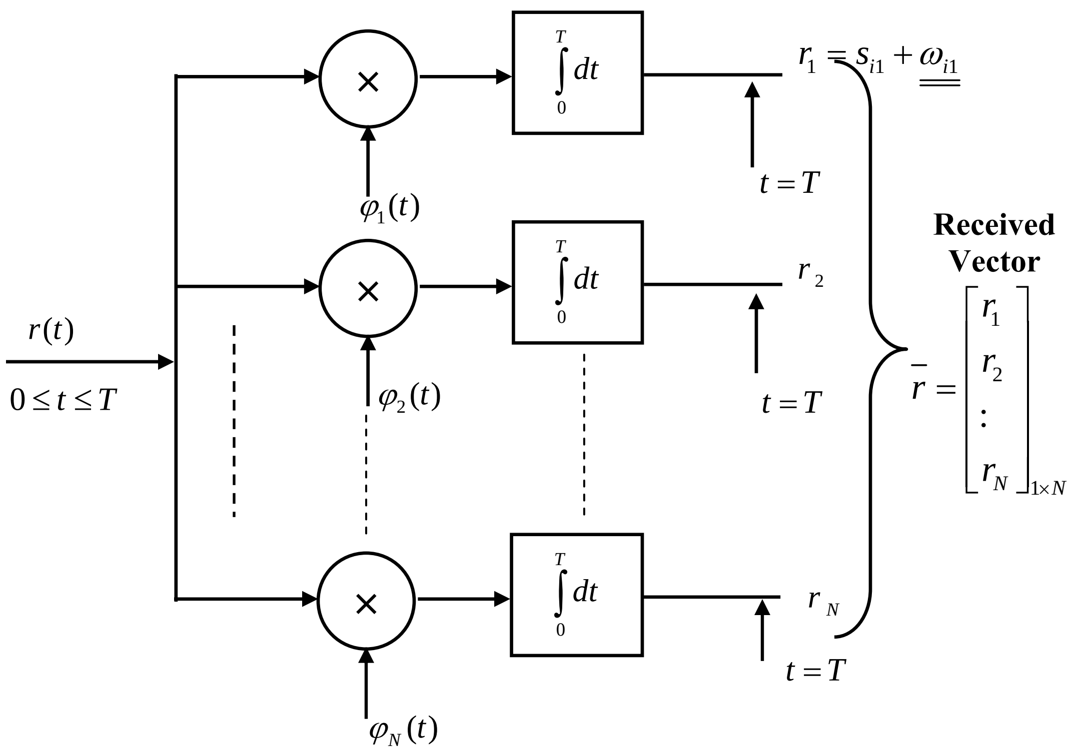

Fig. 4.19.3 *The structure of a Correlation Detector for determining the received vector* $\bar{r}$ *from the received signal r(t)*

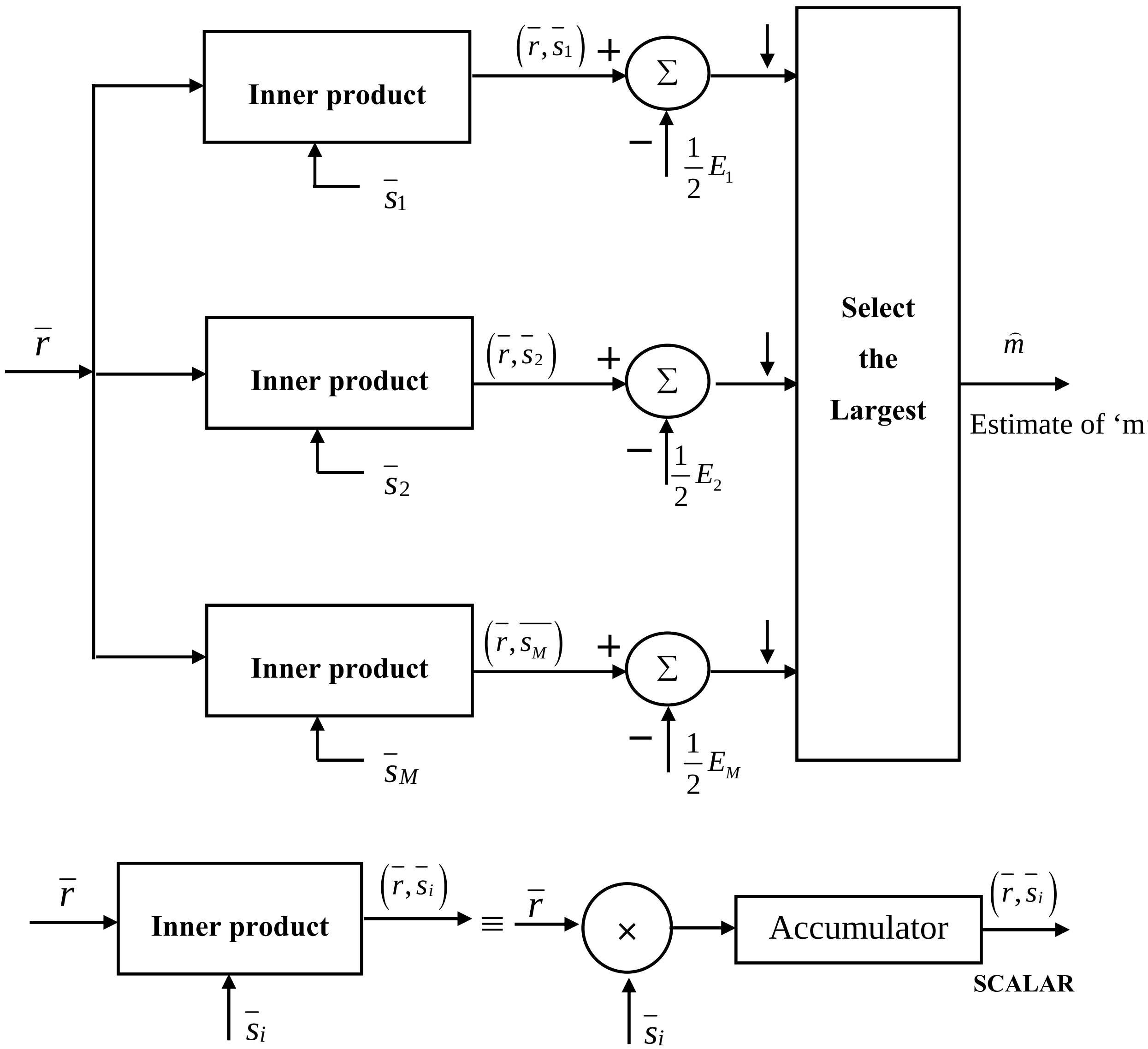

Fig. 4. 19.4 *Block schematic diagram for the Vector Receiver*

Features of the received vector $\bar{r}$

We will now discuss briefly about the statistical features of the received vector $\bar{r}$ as obtained at the output of the correlation detector [**Fig. 4.19.3**]. The j-th element of $\bar{r}$, which is obtained at the output of the j-th correlator once in T second, can be expressed as:

$$r_j = \int_0^T r(t)\Phi_j(t)dt = \int_0^T \left[s_i(t) + w(t)\right]\Phi_j(t)dt$$

$$= s_{ij} + w_j \; ; \quad \text{j=1,2,….., N} \qquad 4.19.9$$

Here w_j is a Gaussian distributed random variable with zero mean and s_{ij} is a scalar signal component of $\bar{s}_i$. Now, the mean of the correlator out put is, $E\left[r_j\right] = E\left[s_{ij} + w_j\right] = E\left[s_{ij}\right] = s_{ij} = m_{rj}$, say. We note that the mean of the correlator out put is independent of the noise process. However, the variances of the correlator outputs are dependent on the strength of accompanying noise:

$$Var\left[r_j\right] = \sigma_{r_j}^2 = E\left[\left(r_j - s_{ij}\right)^2\right] = E\left[w_j^2\right]$$

$$= E\left[\int_0^T w(t)\Phi_j(t)dt \int_0^T w(u)\Phi_j(u)du\right]$$

$$= E\left[\int_0^T\int_0^T \Phi_j(t)\Phi_j(u).w(t)w(u)dtdu\right]$$

Taking the expectation operation inside, we can write

$$\sigma_{rj}^2 = \int_0^T\int_0^T \Phi_j(t)\Phi_j(u)E\left[w(t).w(u)\right]dtdu$$

$$= \int_0^T\int_0^T \Phi_j(t)\Phi_j(u)R_w(t,u)dtdu \qquad 4.19.10$$

Here, R_w (t-u) is the auto correlation of the noise process. As we have learnt earlier, additive white Gaussian noise process is a WSS random process and hence the auto-correlation function may be expressed as, $R_w(t,u) = R_w(t-u)$ and further, $R_w(t-u) = \frac{N_0}{2}\delta(t-u)$, where '$N_o$' is the single-sided noise power spectral density in Watt/Hz. So, the variance of the correlator output now reduces to:

$$\sigma_{rj}^2 = \frac{N_0}{2}\int_0^T\int_0^T \Phi_j(t)\Phi_j(u)\delta(t-u)dtdu$$

$$= \frac{N_0}{2}\int_0^T \Phi_j^2(t)dt = \frac{N_0}{2} \qquad 4.19.11$$

It is interesting to note that the variance of the random signals at the out puts of all N correlators are a) same, b) independent of information-bearing signal waveform and c) dependent only on the noise psd.

Now, the likelihood function for $s_i(t)$, as introduced earlier in Eq.4.19.3 and the ML decision rule [4.19.5] , can be expressed in terms of the output of the correlation detector. The likelihood function for 'm_i' = $\Pr\left(\bar{r}\,|m_i\right)$ = $f_{\bar{r}}\left(\bar{r}\middle|m_i\right) = f_{\bar{r}}\left(\bar{r}\middle|s_i(t)\right)$, where, $f_r(r|m_i)$ is the conditional pdf of 'r' given 'm_i'.

In our case, $f_{\bar{r}}\left(\bar{r}\middle|m_i\right) = \prod_{j=1}^{N} f_{r_j}\left(r_j\middle|m_i\right)$, i = 1,2,…,M 4.19.12

where, $f_{r_j}\left(r_j\middle|m_i\right)$ is the pdf of a Gaussian random variable with mean s_{ij} & var. = $\sigma_{r_j}^2$ = $\frac{N_0}{2}$, i.e., $f_{r_j}\left(r_j\middle|m_i\right) = \frac{1}{\sqrt{2\pi\sigma_{r_j}^2}}.e^{-\frac{(r_j - s_{ij})^2}{2\sigma_{rj}^2}}$ 4.19.13

Combining Eq. 4.19.12 and 4.19.13, we finally obtain,

$$f_{\bar{r}}\left(\bar{r}\middle|m_i\right) = \left(\pi N_0\right)^{-\frac{N}{2}}.\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(r_j - s_{ij}\right)^2\right], \quad i=1,2,\ldots\ldots..,M \qquad 4.19.14$$

This generic expression is of fundamental importance in analyzing error performance of digital modulation schemes [Module #5].

Problems

Q4.19.1) Consider a binary transmission scheme where a bit ‘1’ is represented by +1.0 and a bit ‘0’ is represented by –1.0. Determine the basis function if no carrier modulation scheme is used. If the additive noise is a zero mean Gaussian process, determine the mean values of r_1 and r_2 at the output of the correlation detector. Further, determine E_1 and E_2 as per Fig 4.19.4.