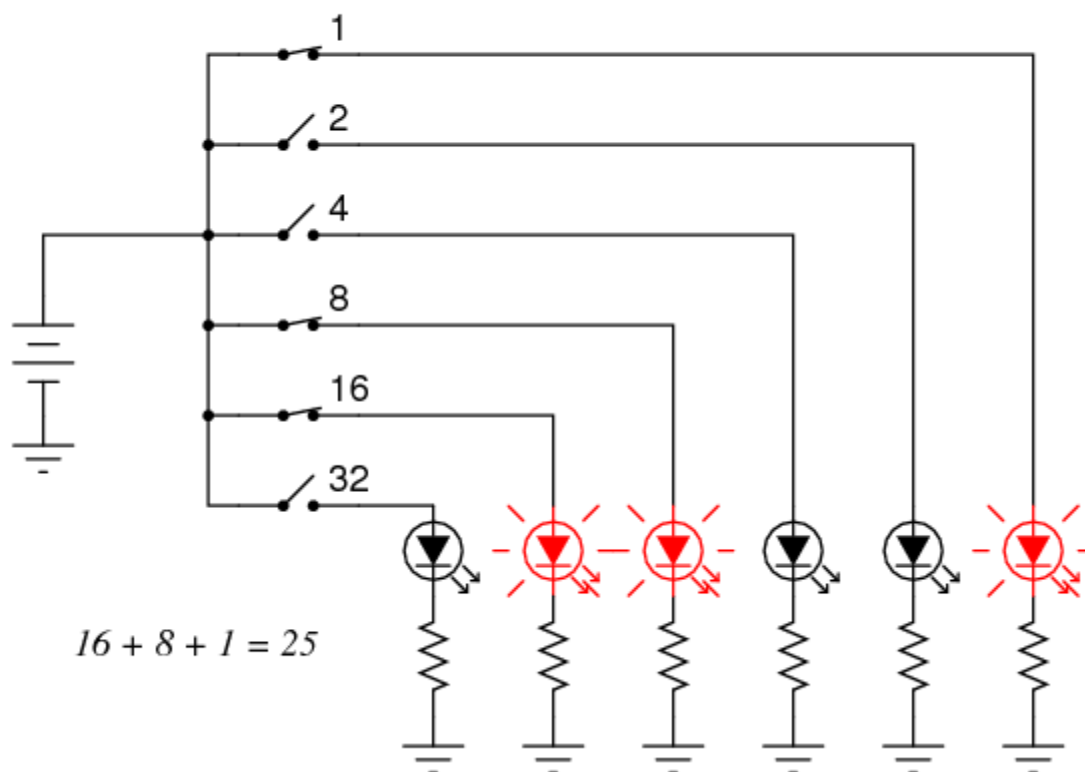


# Computational circuits

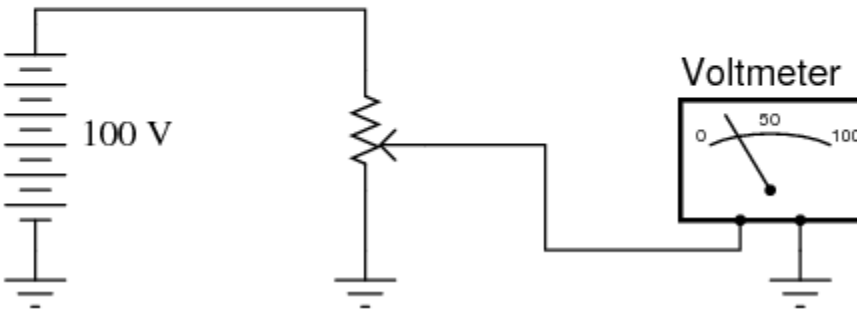
When someone mentions the word "computer," a digital device is what usually comes to mind. Digital circuits represent numerical quantities in *binary* format: patterns of 1's and 0's represented by a multitude of transistor circuits operating in saturated or cutoff states. However, analog circuitry may also be used to represent numerical quantities and perform mathematical calculations, by using variable voltage signals instead of discrete on/off states.

Here is a simple example of binary (digital) representation versus analog representation of the number "twenty-five:"

## A digital circuit representing the number 25:



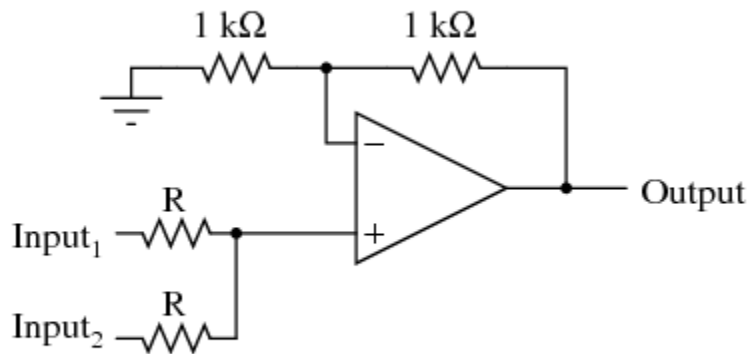
## An analog circuit representing the number 25:



Digital circuits are very different from circuits built on analog principles. Digital computational circuits can be incredibly complex, and calculations must often be performed in sequential "steps" to obtain a final answer, much as a human being would perform arithmetical calculations in steps with pencil and paper. Analog computational circuits, on the other hand, are quite simple in comparison, and perform their calculations in continuous, real-time fashion. There is a disadvantage to using analog circuitry to represent numbers, though: imprecision. The digital circuit shown above is representing the number twenty-five, precisely. The analog circuit shown above may or may not be exactly calibrated to 25.000 volts, but is subject to "drift" and error.

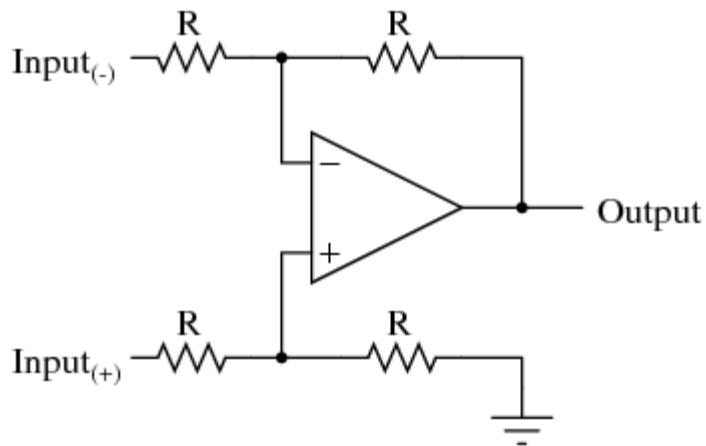
In applications where precision is not critical, analog computational circuits are very practical and elegant. Shown here are a few op-amp circuits for performing analog computation:

*Analog summer (adder) circuit*



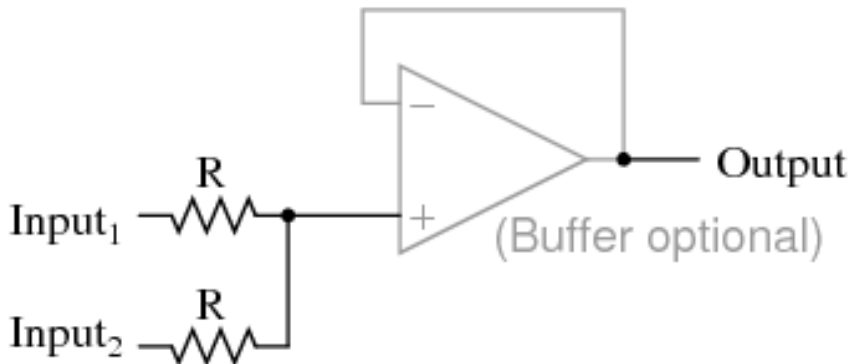
$$\text{Output} = \text{Input}_1 + \text{Input}_2$$

*Analog subtractor circuit*



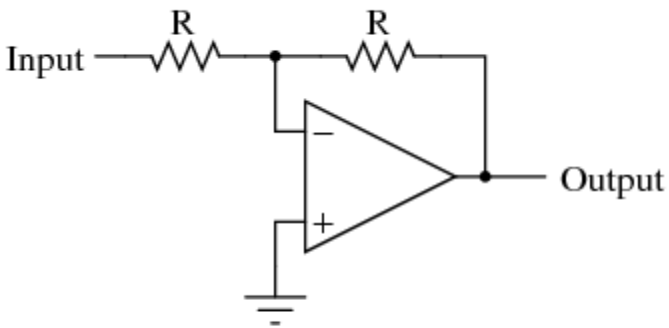
$$\text{Output} = \text{Input}_{(+)} - \text{Input}_{(-)}$$

### *Analog averager circuit*



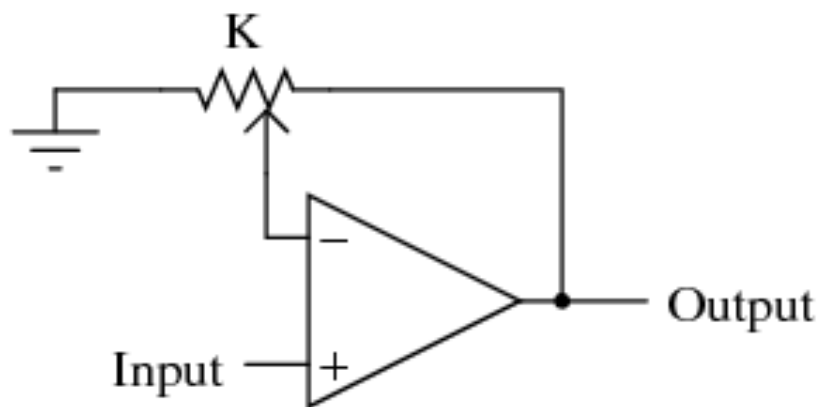
$$\text{Output} = \frac{\text{Input}_1 + \text{Input}_2}{2}$$

### *Analog inverter (sign reverser) circuit*



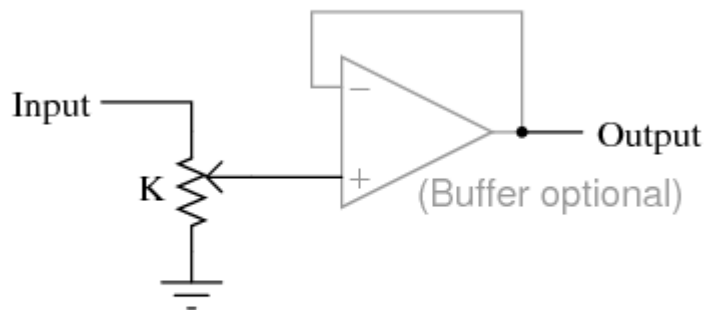
$$\text{Output} = - \text{Input}$$

*Analog "multiply-by-constant" circuit*



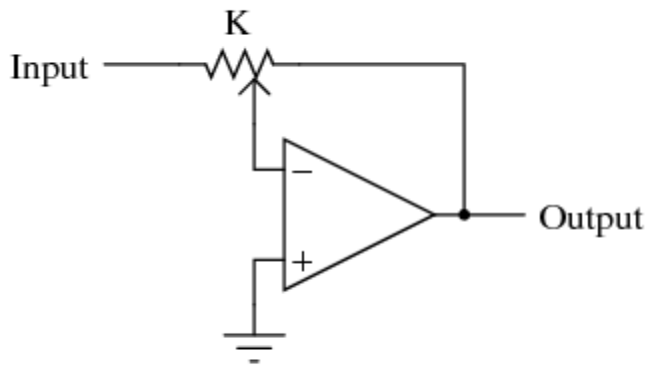
$$\text{Output} = (K)(\text{Input})$$

*Analog "divide-by-constant" circuit*



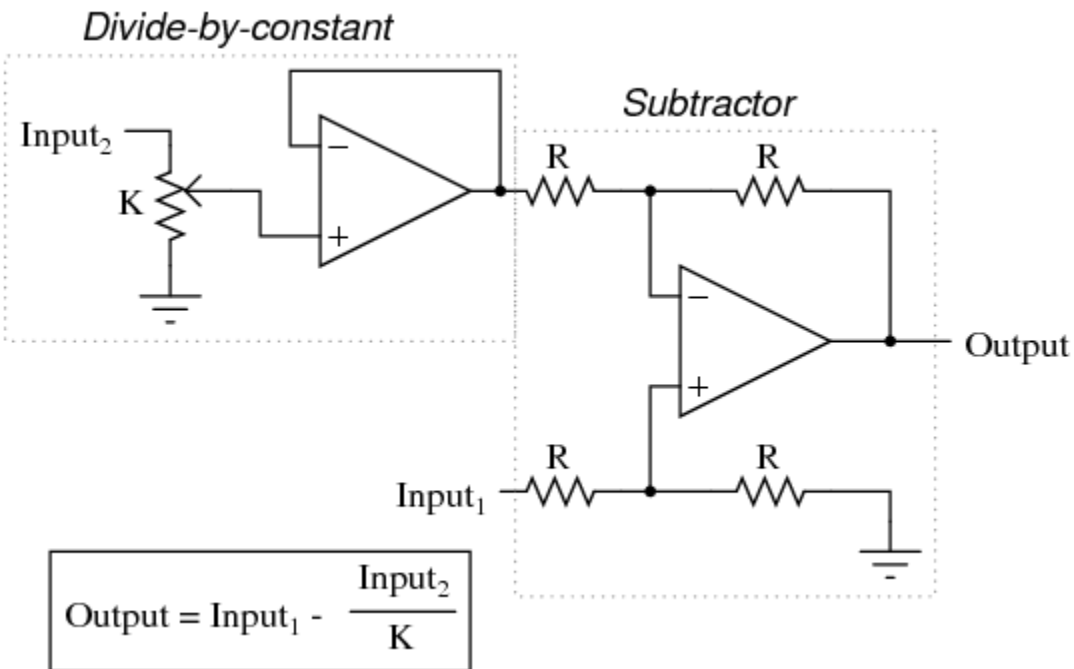
$$\text{Output} = \frac{\text{Input}}{K}$$

*Analog inverting "multiply/divide-by-constant" circuit*



$$\text{Output} = - (K)(\text{Input})$$

Each of these circuits may be used in modular fashion to create a circuit capable of multiple calculations. For instance, suppose that we needed to subtract a certain fraction of one variable from another variable. By combining a divide-by-constant circuit with a subtractor circuit, we could obtain the required function:



$$\text{Output} = \text{Input}_1 - \frac{\text{Input}_2}{K}$$

Devices called *analog computers* used to be common in universities and engineering shops, where dozens of op-amp circuits could be "patched" together with removable jumper wires to model mathematical statements, usually for the purpose of simulating some physical process whose underlying equations were known. Digital computers have made analog computers all but obsolete, but analog computational circuitry cannot be beaten by digital in terms of sheer elegance and economy of necessary components.

Analog computational circuitry excels at performing the calculus operations *integration* and *differentiation* with respect to time, by using capacitors in an op-amp feedback loop. To fully understand these circuits' operation and applications, though, we must first grasp the meaning of these fundamental calculus concepts. Fortunately, the application of op-amp circuits to real-world problems involving calculus serves as an excellent means to teach basic calculus. In the words of John I. Smith, taken from his outstanding textbook, *Modern Operational Circuit Design*:

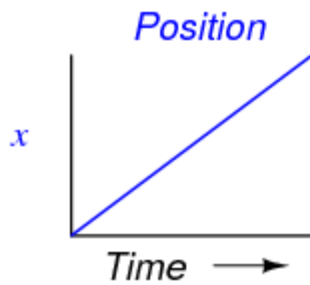
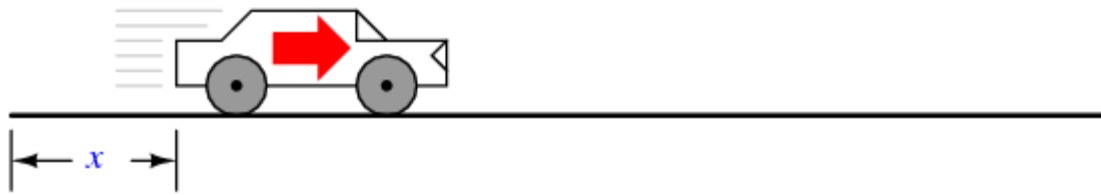
*"A note of encouragement is offered to certain readers: integral calculus is one of the mathematical disciplines that operational [amplifier] circuitry exploits and, in the process, rather demolishes as a barrier to understanding." (pg. 4)*

Mr. Smith's sentiments on the pedagogical value of analog circuitry as a learning tool for mathematics are not unique. Consider the opinion of engineer George Fox Lang, in an article he wrote for the August 2000 issue of the journal *Sound and Vibration*, entitled, "Analog was not a Computer Trademark!":

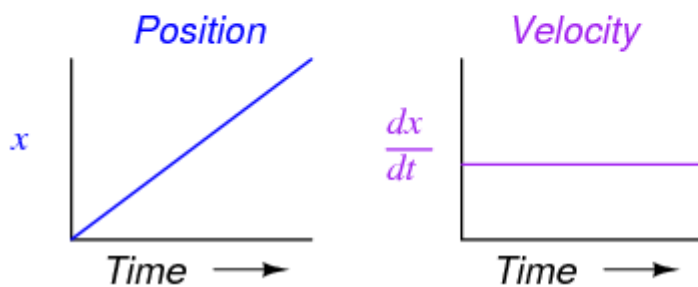
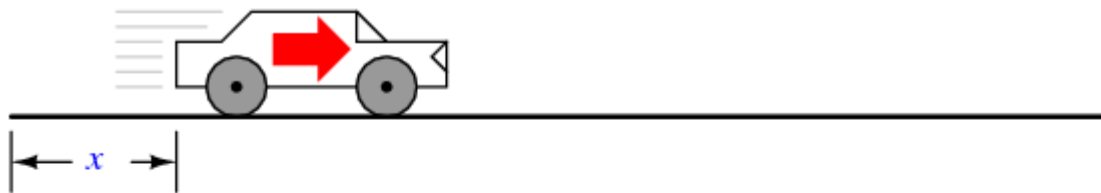
*"Creating a real physical entity (a circuit) governed by a particular set of equations and interacting with it provides unique insight into those mathematical statements. There is no better way to develop a "gut feel" for the interplay between physics and mathematics than to experience such an interaction. The analog computer was a powerful interdisciplinary teaching tool; its obsolescence is mourned by many educators in a variety of fields." (pg. 23)*

Differentiation is the first operation typically learned by beginning calculus students. Simply put, differentiation is determining the instantaneous rate-of-change of one variable as it relates to another. In analog differentiator circuits, the independent variable is time, and so the rates of change we're dealing with are rates of change for an electronic signal (voltage or current) with respect to time.

Suppose we were to measure the position of a car, traveling in a direct path (no turns), from its starting point. Let us call this measurement,  $x$ . If the car moves at a rate such that its distance from "start" increases steadily over time, its position will plot on a graph as a *linear* function (straight line):



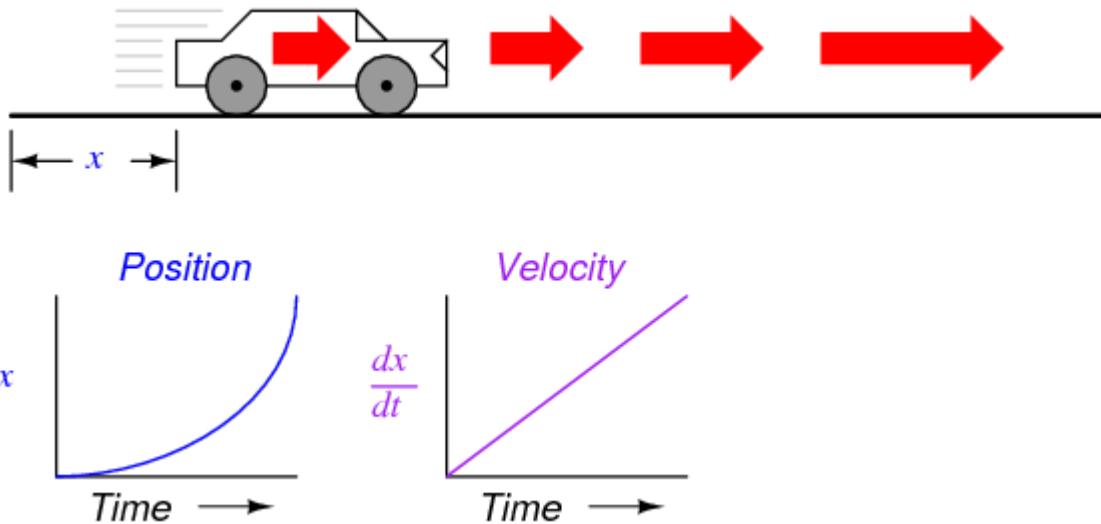
If we were to calculate the *derivative* of the car's position with respect to time (that is, determine the rate-of-change of the car's position with respect to time), we would arrive at a quantity representing the car's velocity. The differentiation function is represented by the fractional notation  $d/d$ , so when differentiating position ( $x$ ) with respect to time ( $t$ ), we denote the result (the derivative) as  $dx/dt$ :



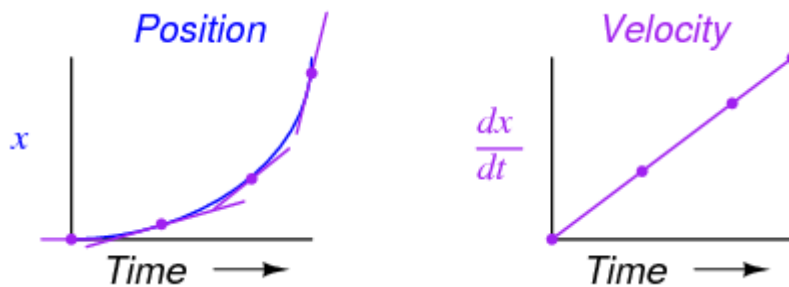
For a linear graph of  $x$  over time, the derivative of position ( $dx/dt$ ), otherwise and more commonly known as *velocity*, will be a flat line, unchanging in value. The derivative of a mathematical function may be graphically understood as its *slope* when plotted on a graph, and here we can see that the position ( $x$ ) graph has a constant slope, which means that its derivative ( $dx/dt$ ) must be constant over time.



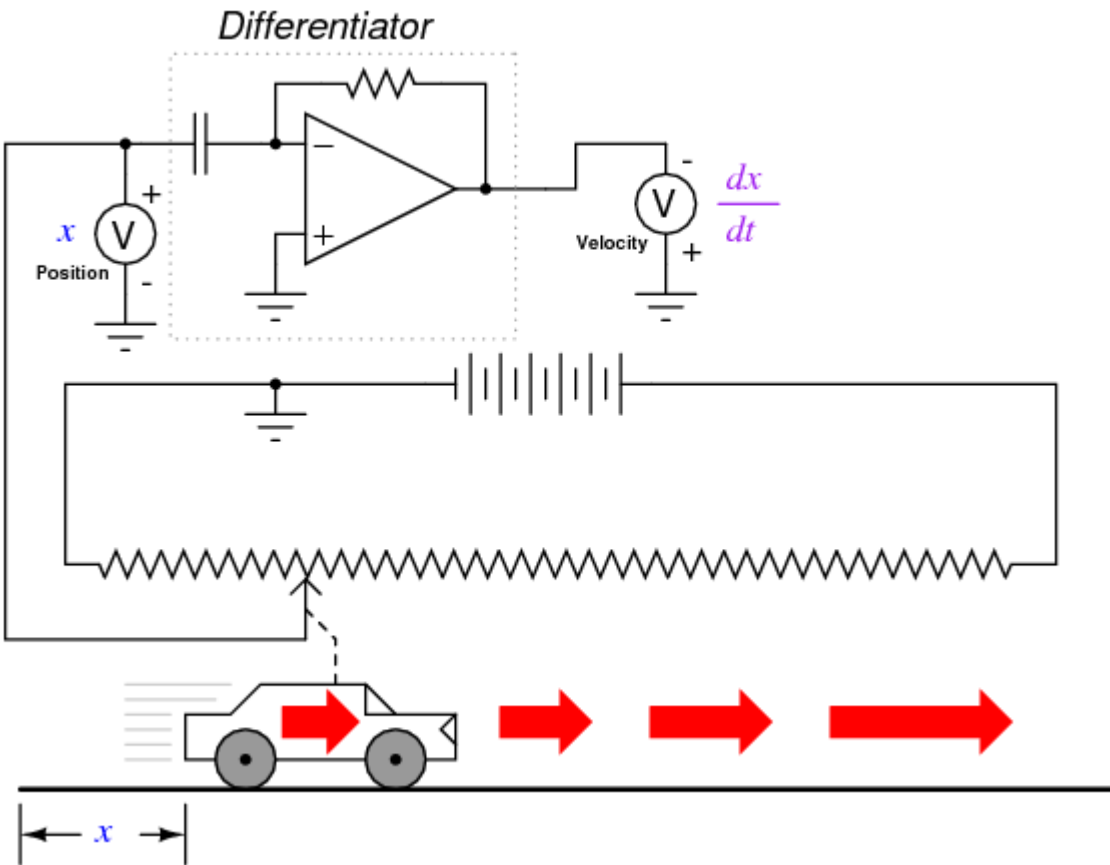
Now, suppose the distance traveled by the car increased exponentially over time: that is, it began its travel in slow movements, but covered more additional distance with each passing period in time. We would then see that the derivative of position ( $dx/dt$ ), otherwise known as velocity ( $v$ ), would not be constant over time, but would increase:



The height of points on the velocity graph correspond to the rates-of-change, or slope, of points at corresponding times on the position graph:



What does this have to do with analog electronic circuits? Well, if we were to have an analog voltage signal represent the car's position (think of a huge potentiometer whose wiper was attached to the car, generating a voltage proportional to the car's position), we could connect a differentiator circuit to this signal and have the circuit continuously *calculate* the car's velocity, displaying the result via a voltmeter connected to the differentiator circuit's output:

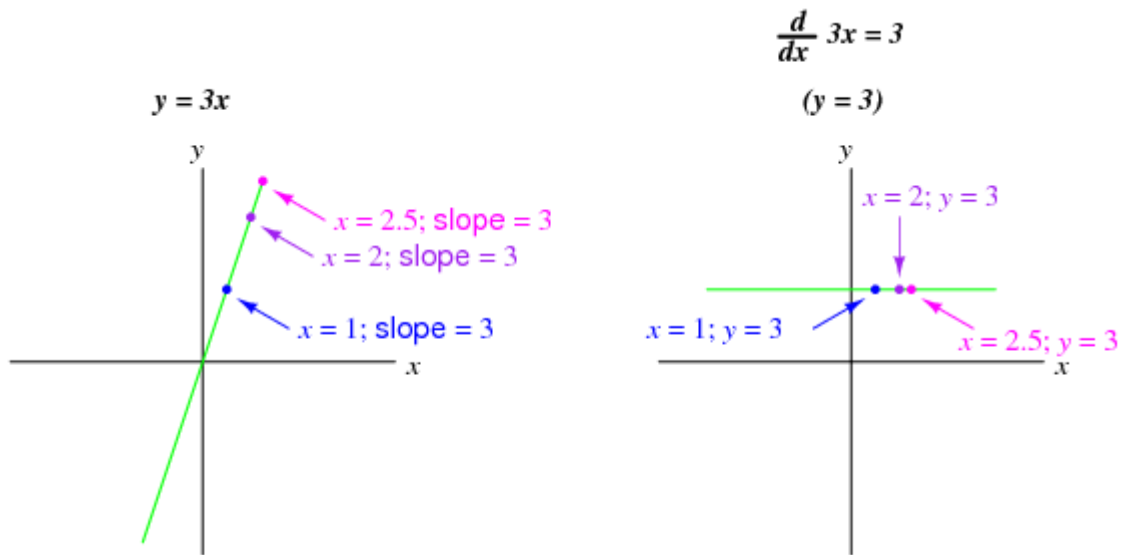


Recall from the last chapter that a differentiator circuit outputs a voltage proportional to the input voltage's *rate-of-change over time* ( $d/dt$ ). Thus, if the input voltage is changing over time at a constant *rate*, the output voltage will be at a constant value. If the car moves in such a way that its elapsed distance over time builds up at a steady rate, then that means the car is traveling at a constant velocity, and the differentiator circuit will output a constant voltage proportional to that velocity. If the car's elapsed distance over time changes in a non-steady manner, the differentiator circuit's output will likewise be non-steady, but always at a level representative of the input's rate-of-change over time.

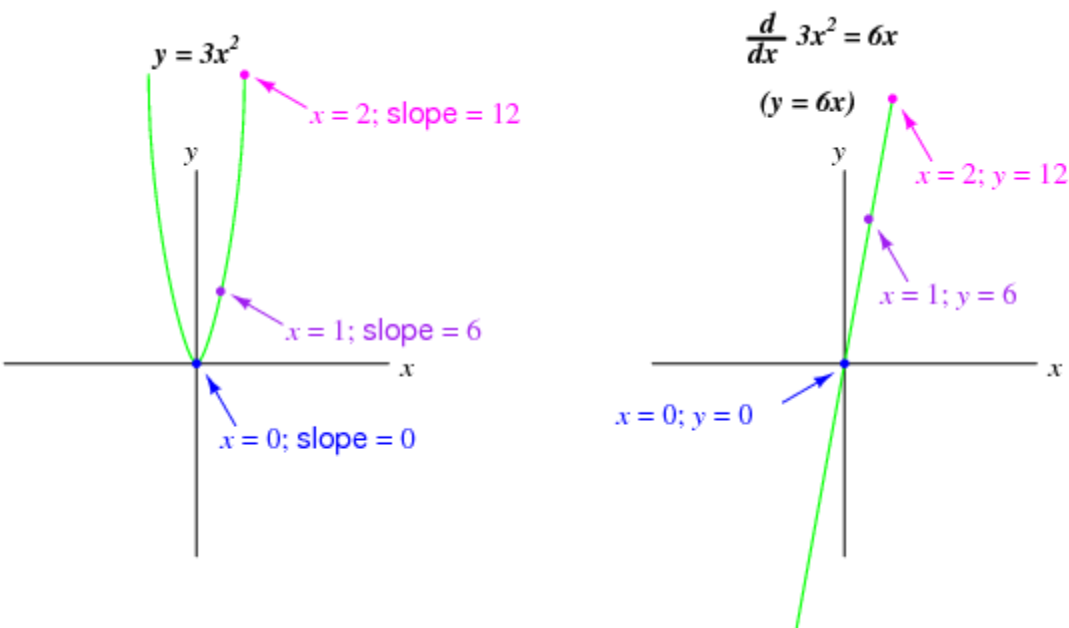
Note that the voltmeter registering velocity (at the output of the differentiator circuit) is connected in "reverse" polarity to the output of the op-amp. This is because the differentiator circuit shown is *inverting*: outputting a negative voltage for a positive input voltage rate-of-change. If we wish to have the voltmeter register a positive value for velocity, it will have to be connected to the op-amp as shown. As impractical as it may be to connect a giant potentiometer to a moving object such as an automobile, the concept should be clear: by electronically

performing the calculus function of differentiation on a signal representing position, we obtain a signal representing velocity.

Beginning calculus students learn symbolic techniques for differentiation. However, this requires that the equation describing the original graph be known. For example, calculus students learn how to take a function such as  $y = 3x$  and find its derivative with respect to  $x$  ( $d/dx$ ),  $3$ , simply by manipulating the equation. We may verify the accuracy of this manipulation by comparing the graphs of the two functions:



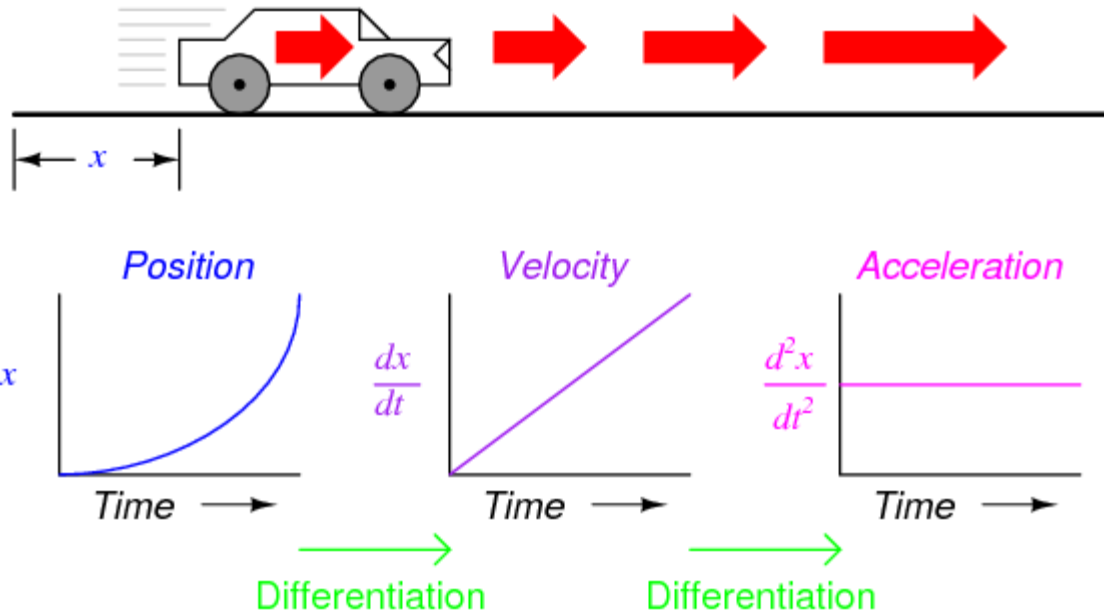
Nonlinear functions such as  $y = 3x^2$  may also be differentiated by symbolic means. In this case, the derivative of  $y = 3x^2$  with respect to  $x$  is  $6x$ :



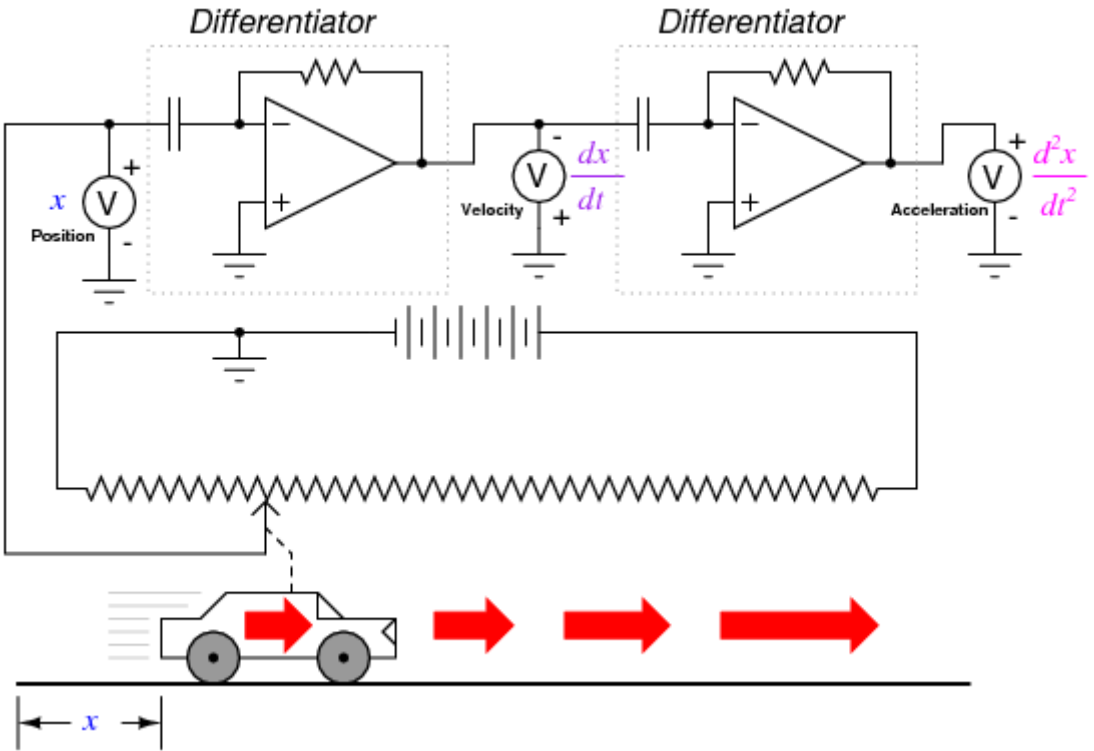
In real life, though, we often cannot describe the behavior of any physical event by a simple equation like  $y = 3x$ , and so symbolic differentiation of the type learned by calculus students may be impossible to apply to a physical measurement. If someone wished to determine the derivative of our hypothetical car's position ( $dx/dt = \text{velocity}$ ) by symbolic means, they would first have to obtain an equation describing the car's position over time, based on position measurements taken from a real experiment -- a nearly impossible task unless the car is operated under carefully controlled conditions leading to a very simple position graph. However, an analog differentiator circuit, by exploiting the behavior of a capacitor with respect to voltage, current, and time  $i = C(dv/dt)$ , naturally differentiates any real signal in relation to time, and would be able to output a signal corresponding to instantaneous velocity ( $dx/dt$ ) at any moment. By logging the car's position signal along with the differentiator's output signal using a chart recorder or other data acquisition device, both graphs would naturally present themselves for inspection and analysis.

We may take the principle of differentiation one step further by applying it to the velocity signal using another differentiator circuit. In other words, use it to calculate the rate-of-change of velocity, which we know is the rate-of-change of position. What practical measure would we arrive at if we did this? Think of this in terms of the units we use to measure position and velocity. If we were to measure the car's position from its starting point in miles, then we would probably express its velocity in units of miles *per hour* ( $dx/dt$ ). If we were to differentiate the velocity (measured in miles per hour) with respect to time, we would end up with a unit of miles per hour *per hour*. Introductory physics classes teach students about the behavior of

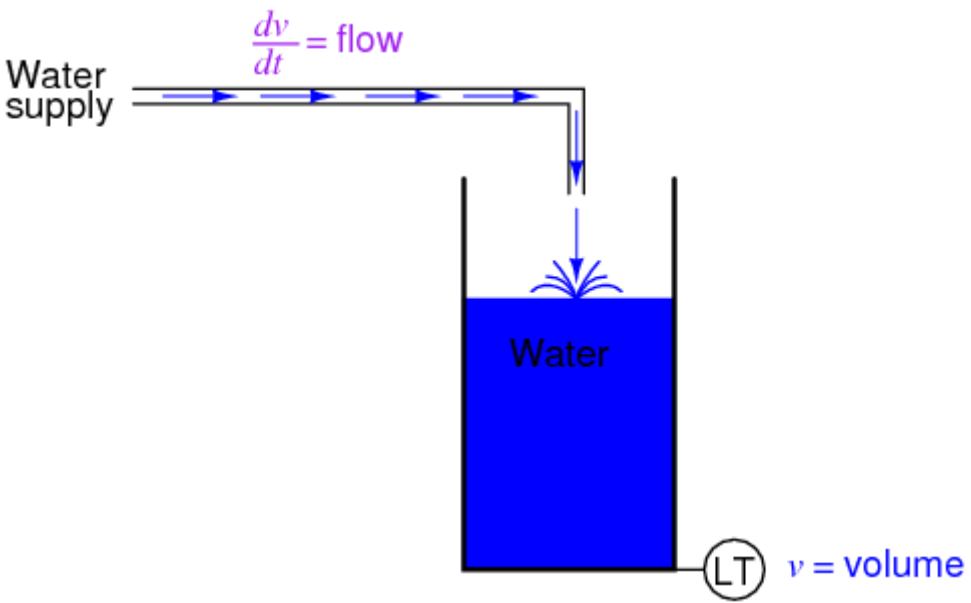
falling objects, measuring position in *meters*, velocity in *meters per second*, and change in velocity over time in *meters per second, per second*. This final measure is called *acceleration*: the rate of change of velocity over time:



The expression  $d^2x/dt^2$  is called the *second derivative* of position ( $x$ ) with regard to time ( $t$ ). If we were to connect a second differentiator circuit to the output of the first, the last voltmeter would register acceleration:

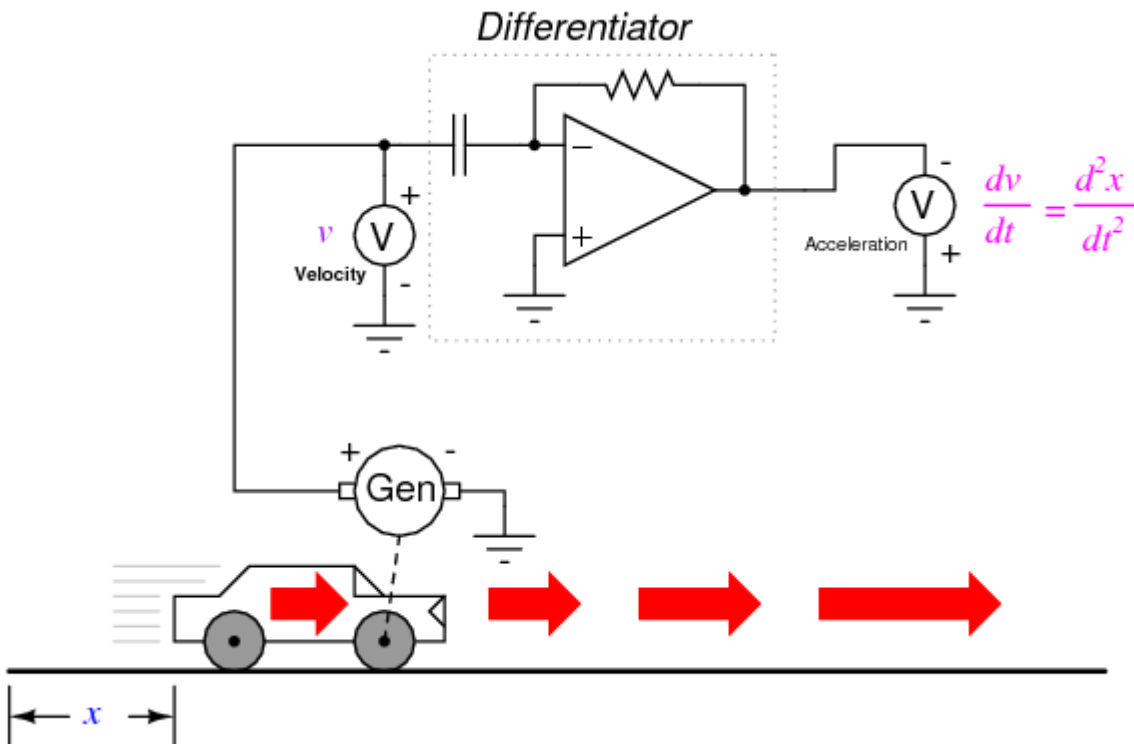


Deriving velocity from position, and acceleration from velocity, we see the principle of differentiation very clearly illustrated. These are not the only physical measurements related to each other in this way, but they are, perhaps, the most common. Another example of calculus in action is the relationship between liquid flow ( $q$ ) and liquid volume ( $v$ ) accumulated in a vessel over time:



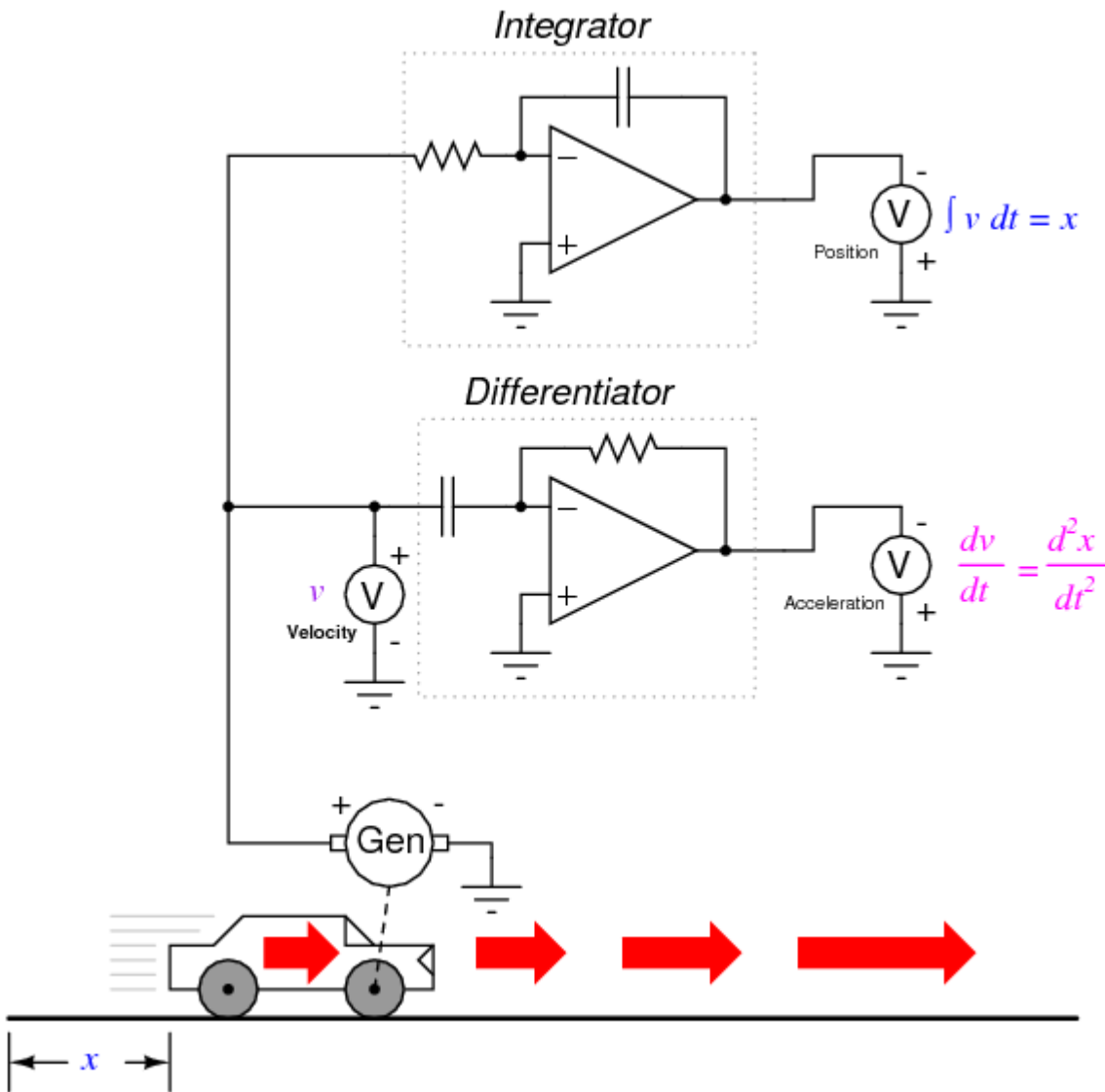
A "Level Transmitter" device mounted on a water storage tank provides a signal directly proportional to water level in the tank, which -- if the tank is of constant cross-sectional area throughout its height -- directly equates water volume stored. If we were to take this volume signal and differentiate it with respect to time ( $dv/dt$ ), we would obtain a signal proportional to the water *flow rate* through the pipe carrying water to the tank. A differentiator circuit connected in such a way as to receive this volume signal would produce an output signal proportional to flow, possibly substituting for a flow-measurement device ("Flow Transmitter") installed in the pipe.

Returning to the car experiment, suppose that our hypothetical car were equipped with a tachogenerator on one of the wheels, producing a voltage signal directly proportional to velocity. We could differentiate the signal to obtain acceleration with one circuit, like this:



By its very nature, the tachogenerator differentiates the car's position with respect to time, generating a voltage proportional to how rapidly the wheel's angular position changes over time. This provides us with a raw signal already representative of velocity, with only a single step of differentiation needed to obtain an acceleration signal. A tachogenerator measuring velocity, of course, is a far more practical example of automobile instrumentation than a giant potentiometer measuring its physical position, but what we gain in practicality we lose in position

measurement. No matter how many times we differentiate, we can never infer the car's position from a velocity signal. If the process of differentiation brought us from position to velocity to acceleration, then somehow we need to perform the "reverse" process of differentiation to go from velocity to position. Such a mathematical process does exist, and it is called *integration*. The "integrator" circuit may be used to perform this function of integration with respect to time:

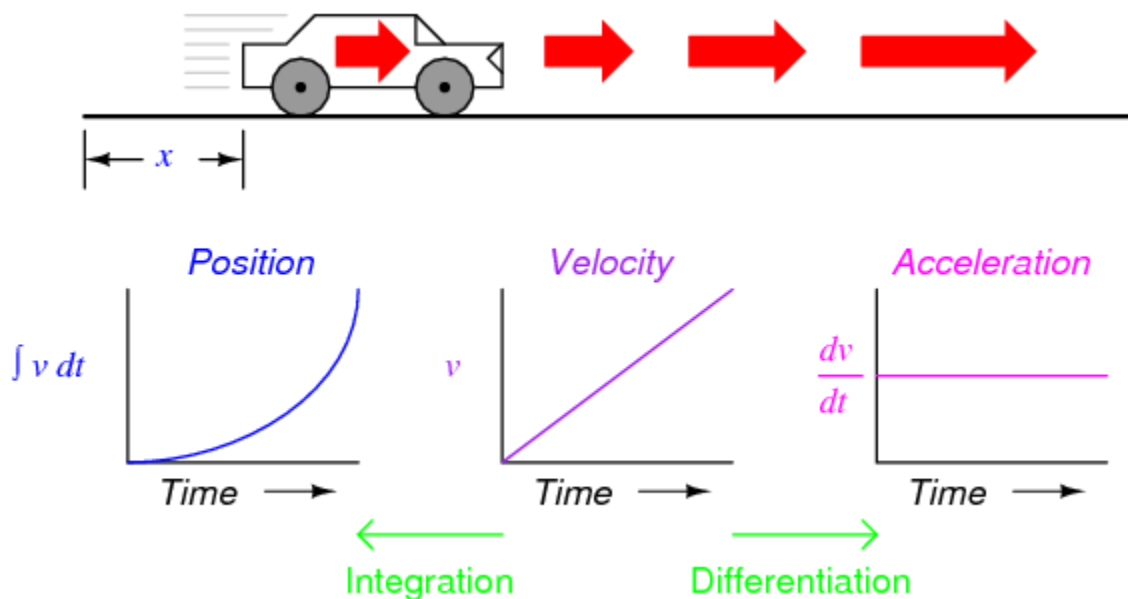


Recall from the last chapter that an integrator circuit outputs a voltage whose rate-of-change over time is proportional to the input voltage's magnitude. Thus, given a constant input voltage, the output voltage will *change* at a constant *rate*. If the car travels at a constant velocity (constant voltage input to the integrator circuit from the tachogenerator), then its distance traveled will increase steadily as time progresses, and the integrator will output a steadily changing voltage proportional

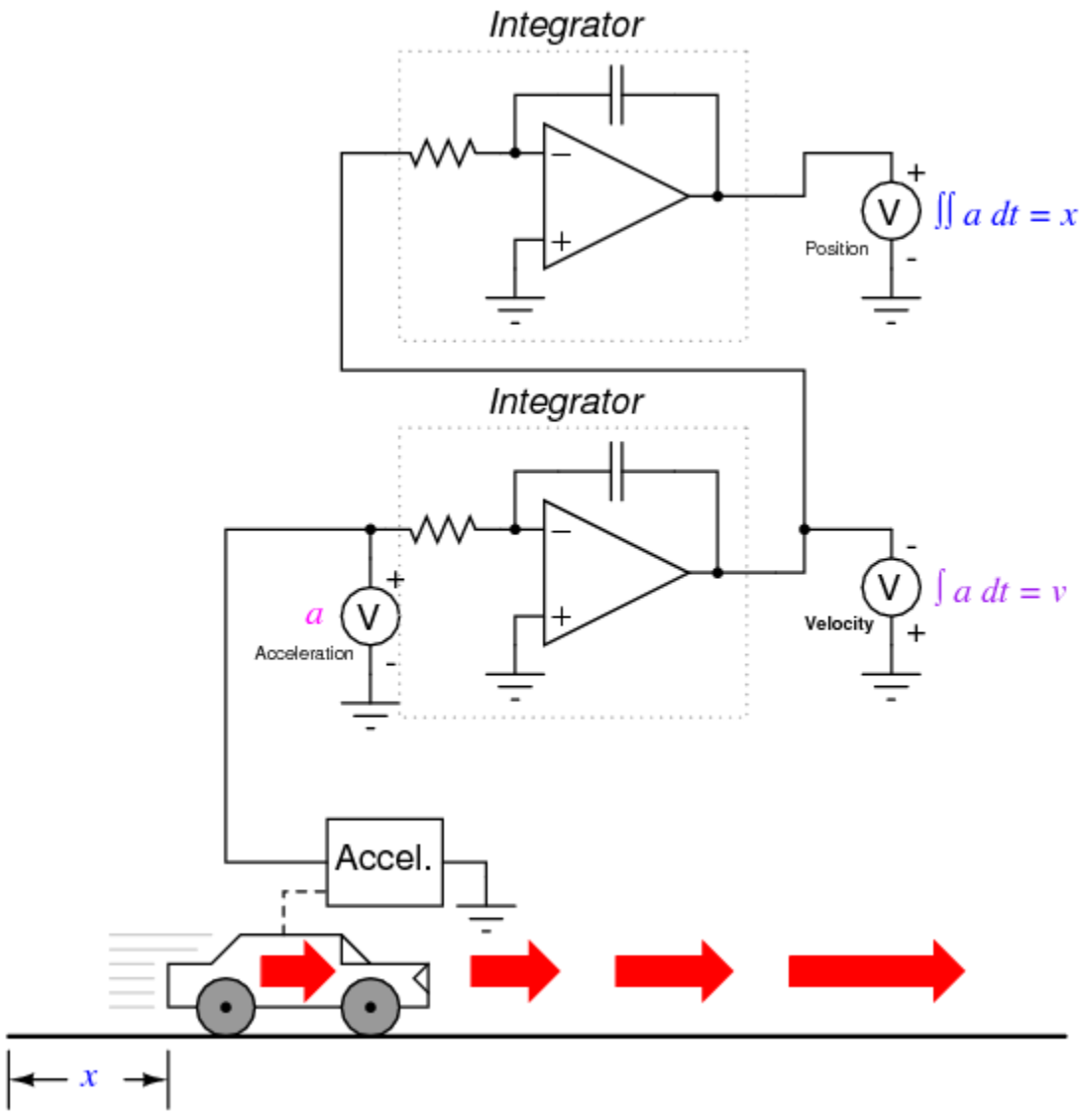


to that distance. If the car's velocity is not constant, then neither will the rate-of-change over time be of the integrator circuit's output, but the output voltage *will* faithfully represent the amount of distance traveled by the car at any given point in time.

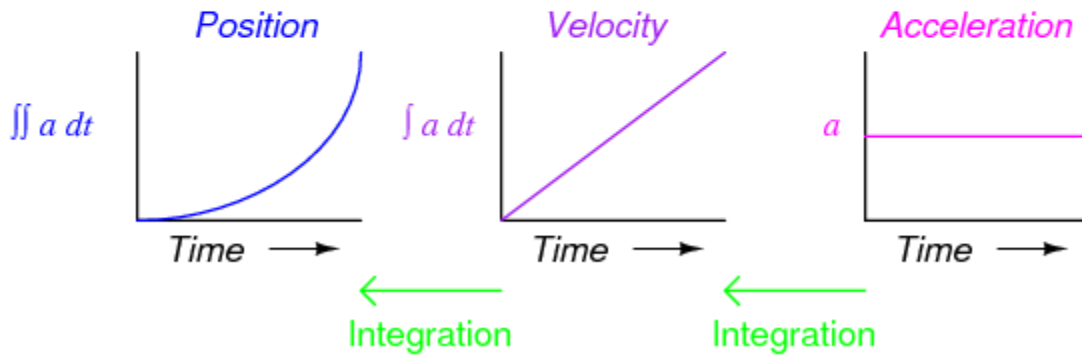
The symbol for integration looks something like a very narrow, cursive letter "S" ( $\int$ ). The equation utilizing this symbol ( $\int v dt = x$ ) tells us that we are integrating velocity ( $v$ ) with respect to time ( $dt$ ), and obtaining position ( $x$ ) as a result. So, we may express three measures of the car's motion (position, velocity, and acceleration) in terms of velocity ( $v$ ) just as easily as we could in terms of position ( $x$ ):



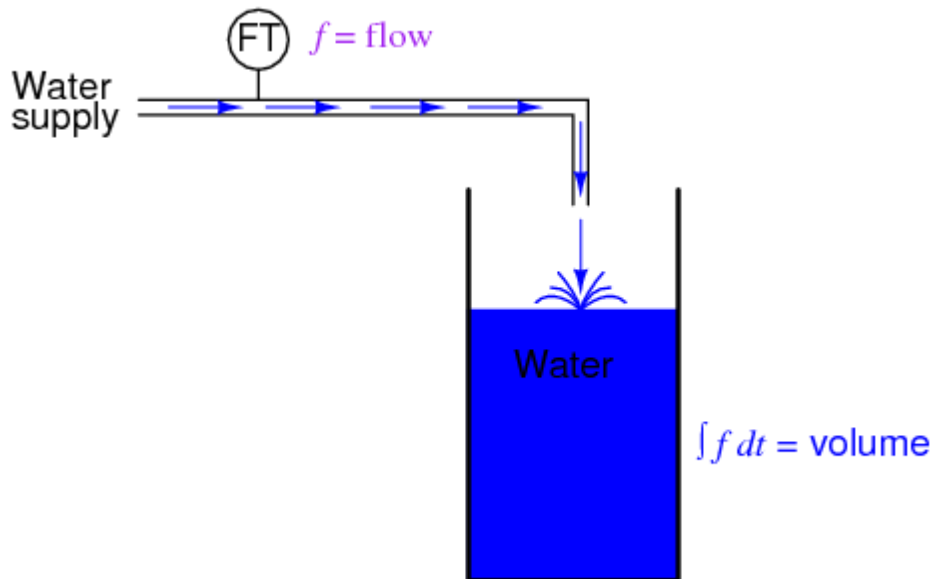
If we had an accelerometer attached to the car, generating a signal proportional to the rate of acceleration or deceleration, we could (hypothetically) obtain a velocity signal with one step of integration, and a position signal with a second step of integration:



Thus, all three measures of the car's motion (position, velocity, and acceleration) may be expressed in terms of acceleration:



As you might have suspected, the process of integration may be illustrated in, and applied to, other physical systems as well. Take for example the water storage tank and flow example shown earlier. If flow rate is the *derivative* of tank volume with respect to time ( $q = dv/dt$ ), then we could also say that volume is the *integral* of flow rate with respect to time:



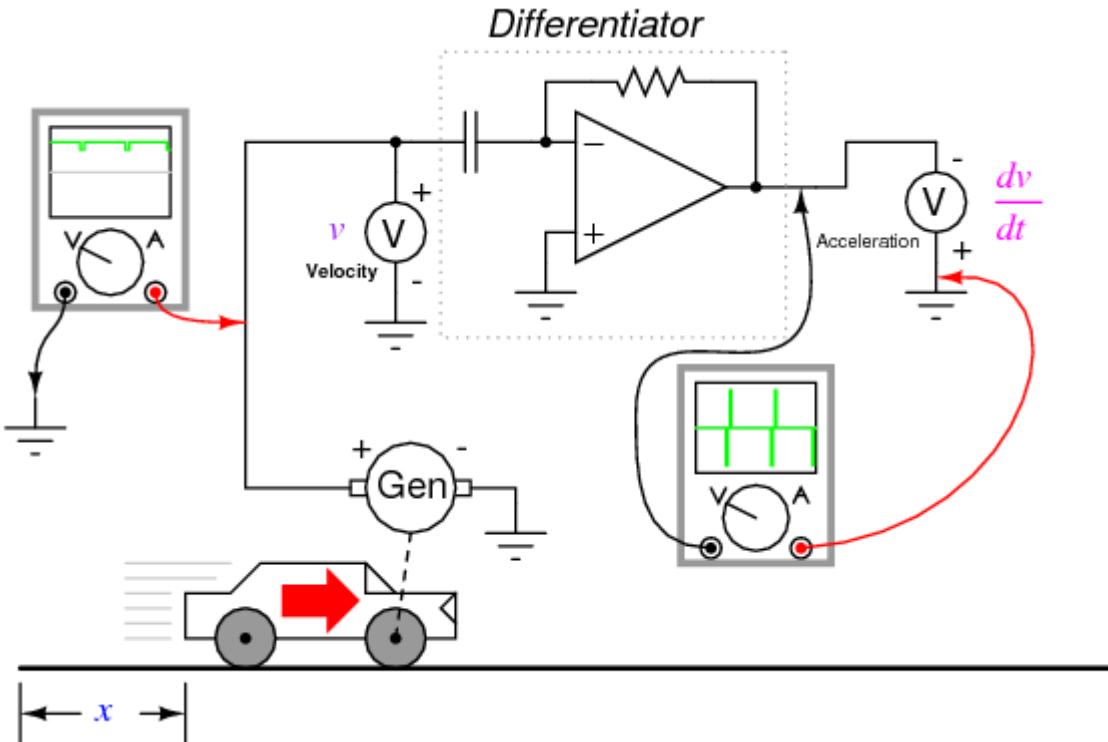
If we were to use a "Flow Transmitter" device to measure water flow, then by time-integration we could calculate the volume of water accumulated in the tank over time. Although it is theoretically possible to use a capacitive op-amp integrator circuit to derive a volume signal from a flow signal, mechanical and digital

electronic "integrator" devices are more suitable for integration over long periods of time, and find frequent use in the water treatment and distribution industries.

Just as there are symbolic techniques for differentiation, there are also symbolic techniques for integration, although they tend to be more complex and varied. Applying symbolic integration to a real-world problem like the acceleration of a car, though, is still contingent on the availability of an equation precisely describing the measured signal -- often a difficult or impossible thing to derive from measured data. However, electronic integrator circuits perform this mathematical function continuously, in real time, and for *any* input signal profile, thus providing a powerful tool for scientists and engineers.

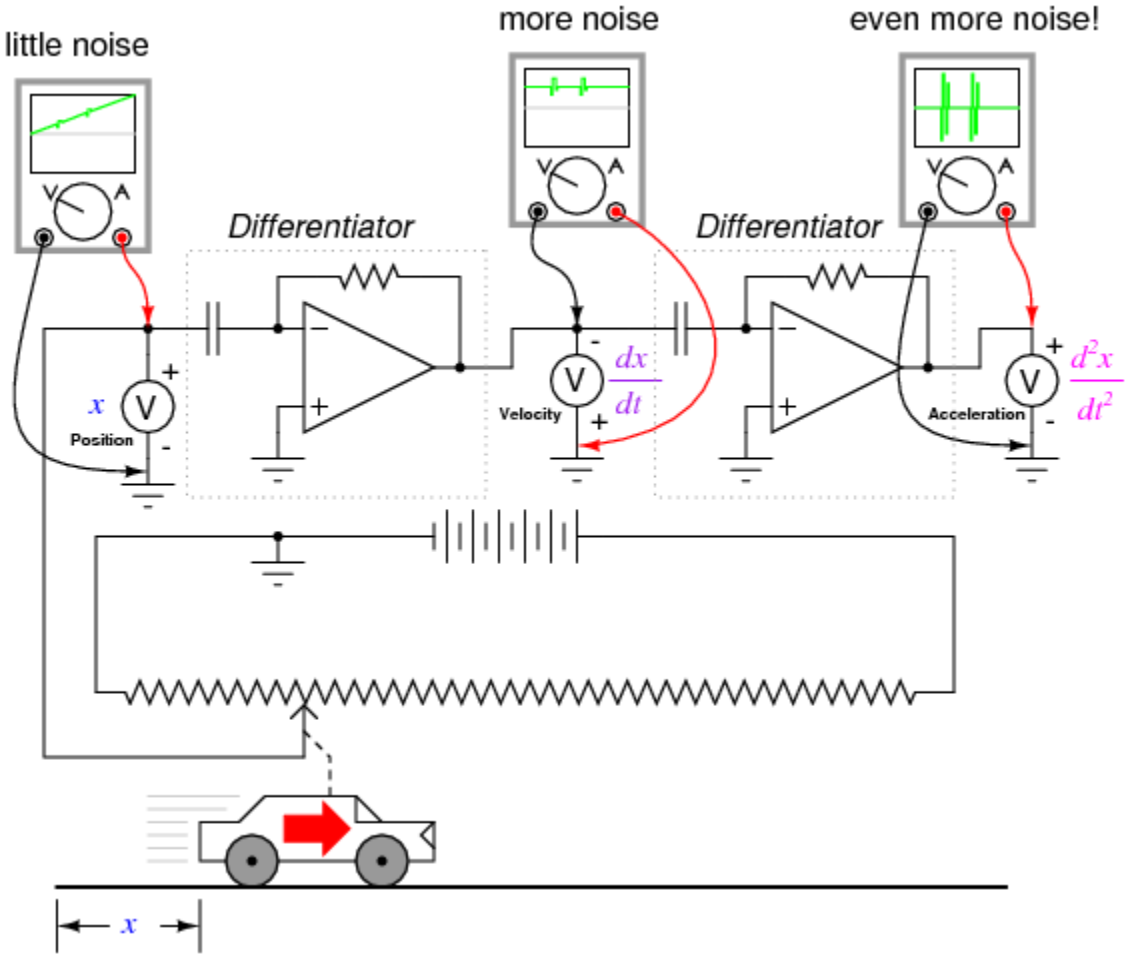
Having said this, there are caveats to the using calculus techniques to derive one type of measurement from another. Differentiation has the undesirable tendency of amplifying "noise" found in the measured variable, since the noise will typically appear as frequencies much higher than the measured variable, and high frequencies by their very nature possess high rates-of-change over time.

To illustrate this problem, suppose we were deriving a measurement of car acceleration from the velocity signal obtained from a tachogenerator with worn brushes or commutator bars. Points of poor contact between brush and commutator will produce momentary "dips" in the tachogenerator's output voltage, and the differentiator circuit connected to it will interpret these dips as very rapid changes in velocity. For a car moving at constant speed -- neither accelerating nor decelerating -- the acceleration signal should be 0 volts, but "noise" in the velocity signal caused by a faulty tachogenerator will cause the differentiated (acceleration) signal to contain "spikes," falsely indicating brief periods of high acceleration and deceleration:

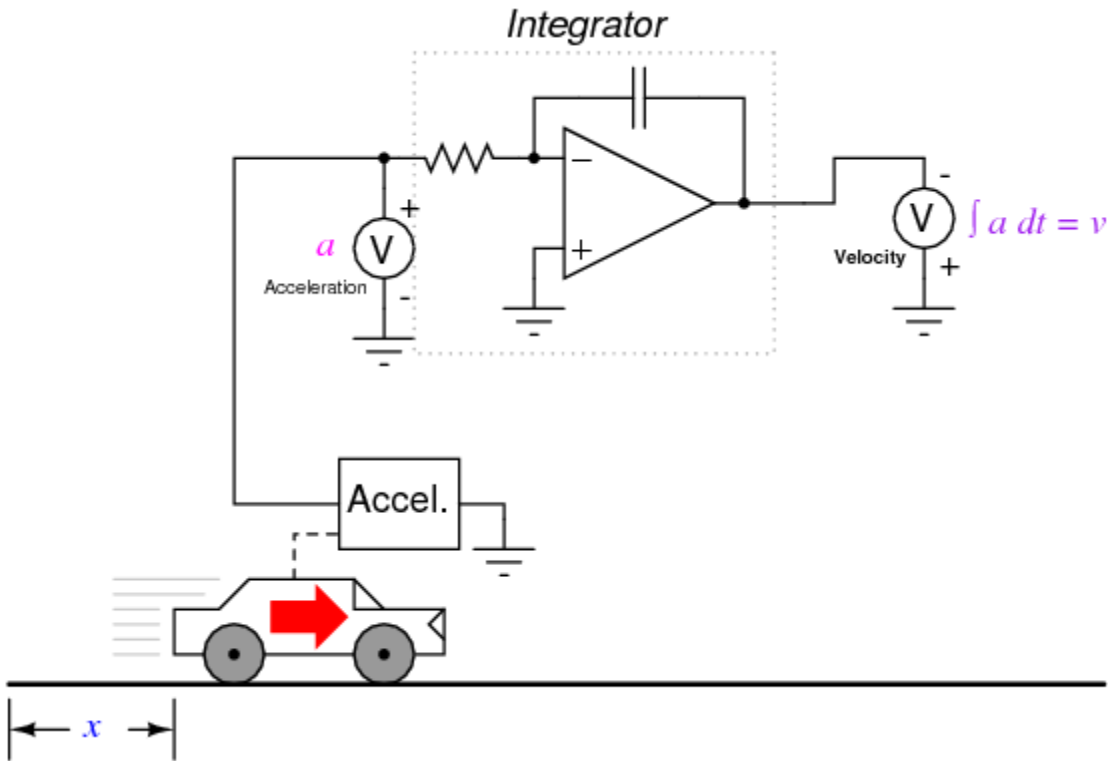


Noise voltage present in a signal to be differentiated need not be of significant amplitude to cause trouble: all that is required is that the noise profile have fast rise or fall times. In other words, any electrical noise with a high  $dv/dt$  component will be problematic when differentiated, even if it is of low amplitude.

It should be noted that this problem is not an artifact (an idiosyncratic error of the measuring/computing instrument) of the analog circuitry; rather, it is inherent to the process of differentiation. No matter how we might perform the differentiation, "noise" in the velocity signal will invariably corrupt the output signal. Of course, if we were differentiating a signal twice, as we did to obtain both velocity and acceleration from a position signal, the amplified noise signal output by the first differentiator circuit will be amplified again by the next differentiator, thus compounding the problem:

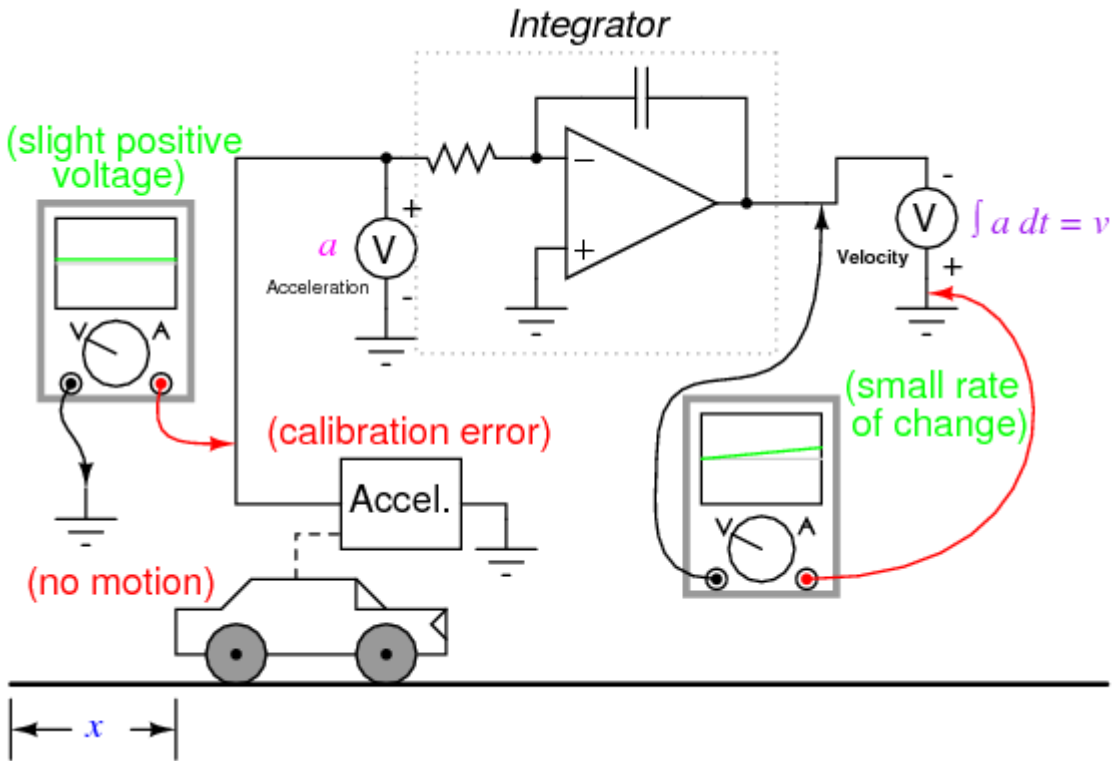


Integration does not suffer from this problem, because integrators act as low-pass filters, attenuating high-frequency input signals. In effect, all the high and low peaks resulting from noise on the signal become averaged together over time, for a diminished net result. One might suppose, then, that we could avoid all trouble by measuring acceleration directly and integrating that signal to obtain velocity; in effect, calculating in "reverse" from the way shown previously:



Unfortunately, following this methodology might lead us into other difficulties, one being a common artifact of analog integrator circuits known as *drift*. All op-amps have some amount of input bias current, and this current will tend to cause a charge to accumulate on the capacitor in addition to whatever charge accumulates as a result of the input voltage signal. In other words, all analog integrator circuits suffer from the tendency of having their output voltage "drift" or "creep" even when there is absolutely no voltage input, accumulating error over time as a result. Also, imperfect capacitors will tend to lose their stored charge over time due to internal resistance, resulting in "drift" toward zero output voltage. These problems *are* artifacts of the analog circuitry, and may be eliminated through the use of digital computation.

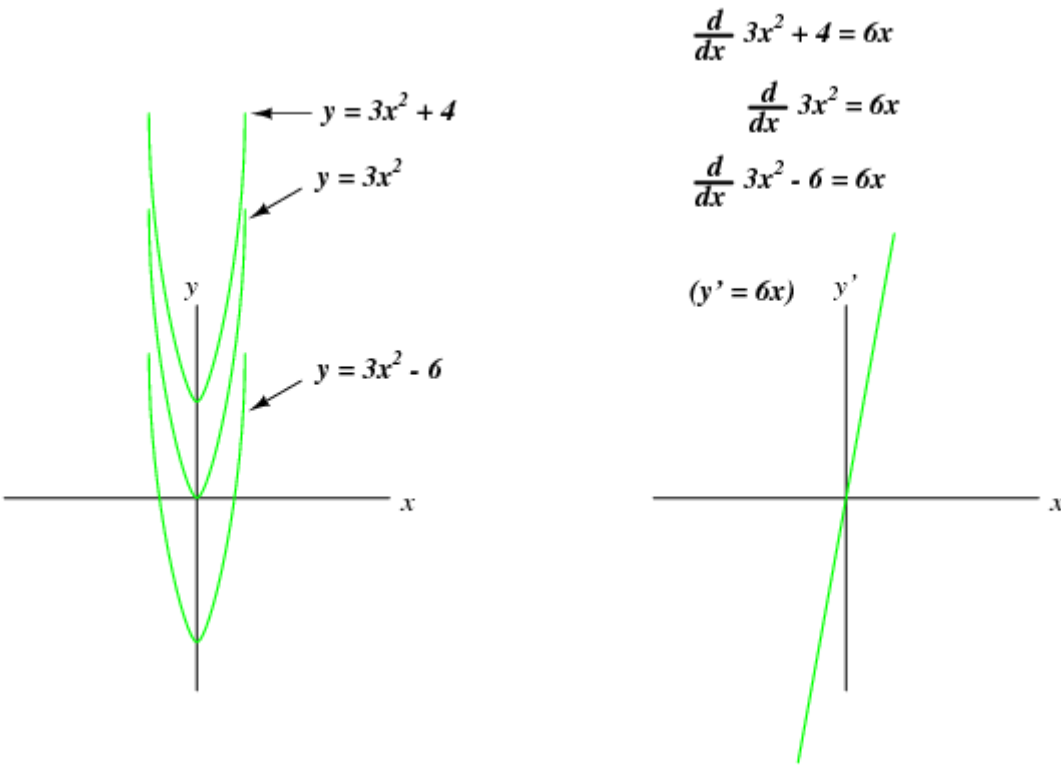
Circuit artifacts notwithstanding, possible errors may result from the integration of one measurement (such as acceleration) to obtain another (such as velocity) simply because of the way integration works. If the "zero" calibration point of the raw signal sensor is not perfect, it will output a slight positive or negative signal even in conditions when it should output nothing. Consider a car with an imperfectly calibrated accelerometer, or one that is influenced by gravity to detect a slight acceleration unrelated to car motion. Even with a perfect integrating computer, this sensor error will cause the integrator to accumulate error, resulting in an output signal indicating a change of velocity when the car is neither accelerating nor decelerating.



As with differentiation, this error will also compound itself if the integrated signal is passed on to another integrator circuit, since the "drifting" output of the first integrator will very soon present a significant positive or negative signal for the next integrator to integrate. Therefore, care should be taken when integrating sensor signals: if the "zero" adjustment of the sensor is not *perfect*, the integrated result will drift, even if the integrator circuit itself is perfect.

So far, the only integration errors discussed have been artificial in nature: originating from imperfections in the circuitry and sensors. There also exists a source of error inherent to the process of integration itself, and that is the *unknown constant* problem. Beginning calculus students learn that whenever a function is integrated, there exists an unknown constant (usually represented as the variable  $C$ ) added to the result. This uncertainty is easiest to understand by comparing the derivatives of several functions differing only by the addition of a constant value:



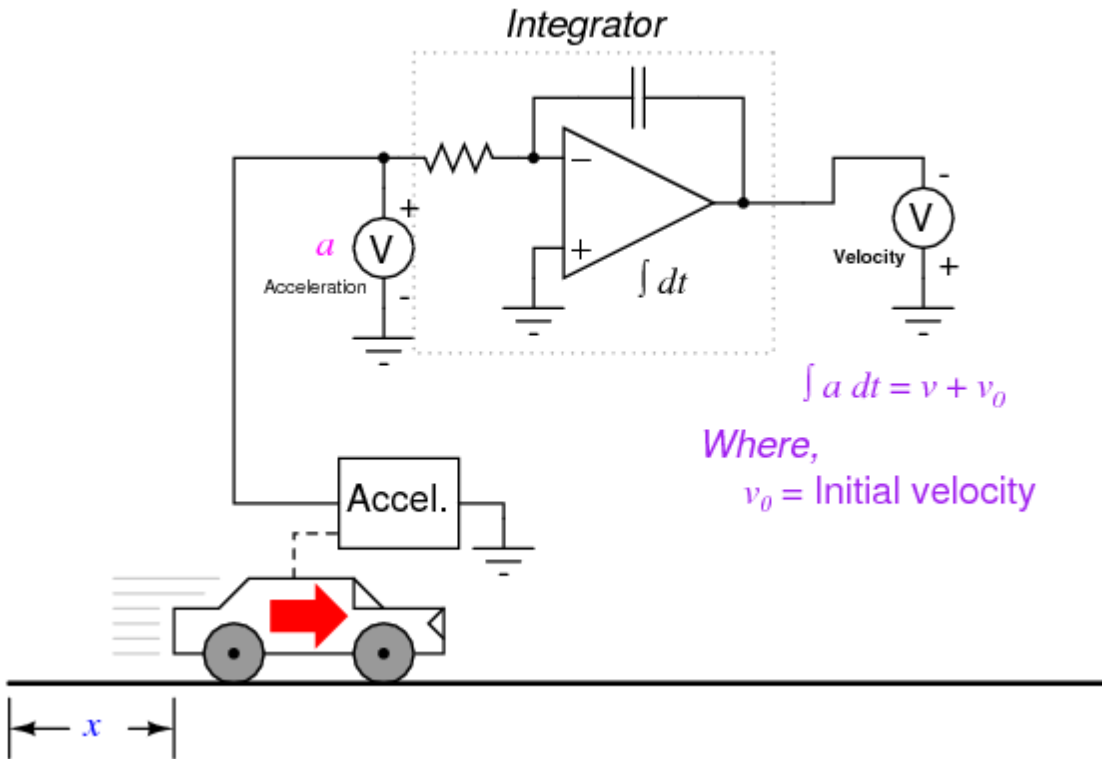


Note how each of the parabolic curves ( $y = 3x^2 + C$ ) share the exact same shape, differing from each other in regard to their vertical offset. However, they all share the exact same derivative function:  $y' = (d/dx)(3x^2 + C) = 6x$ , because they all share identical *rates of change* (slopes) at corresponding points along the  $x$  axis. While this seems quite natural and expected from the perspective of differentiation (different equations sharing a common derivative), it usually strikes beginning students as odd from the perspective of integration, because there are multiple correct answers for the integral of a function. Going from an equation to its derivative, there is only one answer, but going from that derivative back to the original equation leads us to a range of correct solutions. In honor of this uncertainty, the symbolic function of integration is called the *indefinite integral*.

When an integrator performs live signal integration with respect to time, the output is the sum of the integrated input signal over time *and* an initial value of arbitrary magnitude, representing the integrator's pre-existing output at the time integration began. For example, if I integrate the velocity of a car driving in a straight line away from a city, calculating that a constant velocity of 50 miles per hour over a time of 2 hours will produce a distance ( $\int v dt$ ) of 100 miles, that does not necessarily mean the car will be 100 miles away from the city after 2 hours. All it tells us is that the car will be 100 miles *further* away from the city after 2 hours of driving. The actual distance from the city after 2 hours of driving depends on how far the car was from the city when integration began. If we do not know this initial

value for distance, we cannot determine the car's exact distance from the city after 2 hours of driving.

This same problem appears when we integrate acceleration with respect to time to obtain velocity:



In this integrator system, the calculated velocity of the car will only be valid if the integrator circuit is *initialized* to an output value of zero when the car is stationary ( $v = 0$ ). Otherwise, the integrator could very well be outputting a non-zero signal for velocity ( $v_0$ ) when the car is stationary, for the accelerometer cannot tell the difference between a stationary state (0 miles per hour) and a state of constant velocity (say, 60 miles per hour, unchanging). This uncertainty in integrator output is inherent to the process of integration, and not an artifact of the circuitry or of the sensor.

In summary, if maximum accuracy is desired for any physical measurement, it is best to measure that variable directly rather than compute it from other measurements. This is not to say that computation is worthless. Quite to the contrary, often it is the only practical means of obtaining a desired measurement. However, the limits of computation must be understood and respected in order that precise measurements be obtained.

Source: [http://www.allaboutcircuits.com/vol\\_3/chpt\\_9/7.html](http://www.allaboutcircuits.com/vol_3/chpt_9/7.html)