Maxwell's Equations

We have considered Maxwell's curl equations for electrostatic fields and modified for time-varying situations to satisfy Faraday's Law. i.e $\nabla X\overline{E} = -\frac{\partial \overline{B}}{\partial t}$.

We shell now consider Maxwell's curl equation for magnetic fields (Ampere's Law) for time-varying conditions.

For static fields EM fields, we have

$$\nabla X\overline{H} = \overline{J}$$

$\overline{J}$, includes conduction and convection currents.

$$\overline{J} = \sigma\overline{E} + \rho\overline{v}$$

$\sigma\overline{E} =$ Conduction current, due to the presence of the electric filed in the conducting medium,

$\rho\overline{v} =$ Convection current, due to the motion of a free charge distribution.

The equations are:

$$\nabla X\bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \nabla X\bar{H} = \bar{J}$$

$$\nabla.\bar{D} = \rho \qquad \nabla.\bar{B} = 0$$

Although the four Maxwell's equations are consistent, they are not independent. The two divergence equations can be derived from the two curl equations by making use of the Continuity Equation $\nabla.\bar{J} = -\frac{\partial \rho}{\partial t}$ and Continuity Equation can be derived from four Maxwell's Equations.

Consider $\nabla X\bar{H} = \bar{J}$, and $\nabla.\nabla X\bar{H} = \nabla.\bar{J} = 0$,

but the continuity equation is,

$$\nabla.\bar{J} = -\frac{\partial \rho}{\partial t}$$

for time-varying fields. Ampere's Law equation needs modification. Add a term to the Ampere's Law equation:

$$\nabla X\bar{H} = \bar{J} + \bar{J}_d$$

and operate $\nabla.$ on both sides:

$$\nabla.\left(\nabla X\bar{H}\right) = \nabla.\bar{J} + \nabla.\bar{J}_d$$

$$0 = -\frac{\partial \rho}{\partial t} + \nabla . \bar{J}_d$$

Since,

$$\nabla . \bar{D} = \rho$$

$$\nabla . \left(\bar{J}_d - \frac{\partial \bar{D}}{\partial t} \right) = 0$$

Let, $\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$, Displacement current density, A/m^2. The Ampere’s Law equation for time-varying fields takes the form:

$$\nabla X \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

Then four consistent equations,

$$\nabla X \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad Faraday\, Law$$

$$\nabla X \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad Ampere's\ Law$$

$$\nabla . \bar{D} = \rho \quad Gauss\, Law$$

$$\nabla . \bar{B} = 0 \quad Magnetic\ Flux\ Law$$

are known as Maxwell’s equations.

Maxwell's equations, together with

$$\bar{F} = q\left(\bar{E} + \bar{u}\ X\ \bar{B}\right) \quad \textit{Lorentz Force Equation}$$

$$\nabla.\bar{J} + \frac{\partial \rho}{\rho t} = 0 \quad \textit{Continuity Equation}$$

from the foundation of the electromagnetic theory.

INTEGRAL FORMS OF MAXWELL'S EQUATIONS

Consider $\nabla X\ \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ and integrate over an open surface *S* with a contour *C* and apply Stoke's Theorem:

$$\int_S \nabla X\ \bar{E}.\hat{n}ds = -\int_S \frac{\partial \bar{B}}{\partial t}.\hat{n}ds$$

Now consider,

$$\nabla X\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

And integrate over a surface *S*

$$\int_S \nabla X\ \bar{H}.\hat{n}ds = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right).\hat{n}ds$$

$$\oint_C \bar{H}.d\bar{l} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t} \right).\hat{n}ds$$

$$\oint_C \bar{H}.d\bar{l} = I + \int_S \left(\frac{\partial \bar{D}}{\partial t} \right).\hat{n}ds$$

For the divergence equations:

Consider $\nabla.\bar{B} = 0$,

Take the volume integral of both sides of the divergence equation over a volume *V* and apply the divergence theorem:

$$\int_v (\nabla.\bar{B})dv = 0$$

results,

$$\oint_s \bar{B}.\hat{n}ds = 0$$

And similarly, $\nabla.\bar{D} = \rho$

$$\int_v (\nabla.\bar{D})dv = \int_v \rho dv$$

$$\oint_s \bar{D}.d\bar{s} = Q$$

Consequently, the boundary conditions remain valid for the time-varying fields, where $\hat{a}_n$ is the unit vector normal to the boundary.

$$E_{2_t} = E_{1_t} \quad or \quad \left(\bar{E}_1 - \bar{E}_2\right) \mathrm{X} \hat{a}_n = 0$$

$$H_{1_t} - H_{2_t} = J \quad or \quad \left(\bar{H}_1 - \bar{H}_2\right) \mathrm{X} \hat{a}_n = \bar{J}$$

$$D_{1_n} - D_{2_n} = \rho_s \quad or \quad \left(D_1 - D_2\right).\hat{a}_n = \rho_s$$

$$B_{1_n} - B_{2_n} = 0 \quad or \quad \left(B_1 - B_2\right).\hat{a}_n = 0$$

Source: http://opencourses.emu.edu.tr/pluginfile.php/1986/mod_resource/content/0/

s_Equations.pdf