Complex Fourier Series

In an earlier **module**, we showed that a square wave could be expressed as a superposition of pulses. As useful as this decomposition was in this example, it does not generalize well to other periodic signals: How can a superposition of pulses equal a smooth signal like a sinusoid? Because of the importance of sinusoids to linear systems, you might wonder whether they could be added together to represent a large number of periodic signals. You would be right and in good company as well. Euler and Gauss in particular worried about this problem, and Jean Baptiste Fourier got the credit even though tough mathematical issues were not settled until later. They worked on what is now known as the **Fourier series**: representing **any** periodic signal as a superposition of sinusoids.

But the Fourier series goes well beyond being another signal decomposition method. Rather, the Fourier series begins our journey to appreciate how a signal can be described in either the time-domain or the frequency-domain with **no** compromise. Let $s(t)$ be a **periodic** signal with period T. We want to show that periodic signals, even those that have constant-valued segments like a square wave, can be expressed as sum of **harmonically** related sine waves: sinusoids having frequencies that are integer multiples of the **fundamental frequency**. Because the signal has period T, the fundamental frequency is $\frac{1}{T}$. The complex Fourier series expresses the signal as a superposition of complex exponentials having frequencies $\frac{k}{T}, k=\{\ldots,-1,0,1,\ldots\}$.

$$s(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T}}$$

(1)

with $c_k=\frac{1}{2}(a_k-jb_k)$. The real and imaginary parts of the **Fourier coefficients** c_k are written in this unusual way for convenience in defining the classic Fourier series. The zeroth coefficient equals the signal's average value and is real- valued for real-valued signals: $c_0=a_0$. The family of functions $\{e^{j\frac{2\pi kt}{T}}\}$ are called **basis functions** and form the foundation of the Fourier series. No matter what the periodic signal might be, these functions are always present and form the representation's building blocks. They depend on the signal period T, and are indexed by k.

KEY POINT:

Assuming we know the period, knowing the Fourier coefficients is equivalent to knowing the signal. Thus, it makes no difference if we have a time-domain or a frequency- domain characterization of the signal.

To find the Fourier coefficients, we note the orthogonality property

$$\int_0^T e^{j\frac{2\pi kt}{T}} e^{(-j)\frac{2\pi lt}{T}}\,dt=\begin{cases}T & \text{if } k=l\\ 0 & \text{if } k\neq l\end{cases}$$

(3)

Assuming for the moment that the complex Fourier series "works," we can find a signal's complex Fourier coefficients, its **spectrum**, by exploiting the orthogonality properties of harmonically related complex exponentials. Simply multiply each side of **Equation 1** by $e^{-(j2\pi lt)}$ and integrate over the interval $[0,T]$.

$$c_k=\frac{1}{T}\int_0^T s(t)e^{-(j\frac{2\pi kt}{T})}\,dt \qquad c_0=\frac{1}{T}\int_0^T s(t)\,dt$$

(4)

EXAMPLE 1

Finding the Fourier series coefficients for the square wave $\mathrm{sq}_T(t)$ is very simple. Mathematically, this signal can be expressed as

$$\mathrm{sq}_T(t) = \begin{cases} 1 & \text{if } 0 < t < \frac{T}{2} \\ -1 & \text{if } \frac{T}{2} < t < T \end{cases}$$

The expression for the Fourier coefficients has the form

$$c_k = \frac{1}{T}\int_0^{\frac{T}{2}} e^{-\left(\frac{j 2\pi k t}{T}\right)}\, dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} e^{-\left(\frac{j 2\pi k t}{T}\right)}\, dt$$

(5)

NOTE:

When integrating an expression containing j, treat it just like any other constant.

The two integrals are very similar, one equaling the negative of the other. The final expression becomes

$$c_k = \frac{-2}{j 2\pi k}\left((-1)^k - 1\right) = \begin{cases} \frac{2}{j\pi k} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}$$

(6)

$$\mathrm{sq}(t) = \sum_{k \in \{\dots,-3,-1,1,3,\dots\}} \frac{2}{j\pi k} e^{(j)\frac{2\pi k t}{T}}$$

(7)

Consequently, the square wave equals a sum of complex exponentials, but only those having frequencies equal to odd multiples of the fundamental frequency $\frac{1}{T}$. The coefficients decay slowly as the frequency index k increases. This index corresponds to the k-th harmonic of the signal's period.

A signal's Fourier series spectrum c_k has interesting properties.

PROPERTY 1

If $s(t)$ is real, $c_k = c_{-k}^*$ (real-valued periodic signals have conjugate-symmetric spectra).

This result follows from the integral that calculates the c_k from the signal. Furthermore, this result means that $\mathrm{Re}(c_k) = \mathrm{Re}(c_{-k})$: The real part of the Fourier coefficients for real-valued signals is even. Similarly, $\mathrm{Im}(c_k) = -\mathrm{Im}(c_{-k})$: The imaginary parts of the Fourier coefficients have odd symmetry. Consequently, if you are given the Fourier coefficients for positive indices and zero and are told the signal is real-valued, you can find the negative-indexed coefficients, hence the entire spectrum. This kind of symmetry, $c_k = c_{-k}^*$, is known as **conjugate symmetry**.

PROPERTY 2

If $s(-t) = s(t)$, which says the signal has even symmetry about the origin, $c_{-k} = c_k$.

Given the previous property for real-valued signals, the Fourier coefficients of even signals are real-valued. A real-valued Fourier expansion amounts to an expansion in terms of only cosines, which is the simplest example of an even signal.

PROPERTY 3

If $s(-t)=-s(t)$, which says the signal has odd symmetry, $c_{-k}=-c_k$.

Therefore, the Fourier coefficients are purely imaginary. The square wave is a great example of an odd-symmetric signal.

PROPERTY 4

The spectral coefficients for a periodic signal delayed by τ, $s(t-\tau)$, are $c_k e^{-\frac{j2\pi k\tau}{T}}$, where c_k denotes the spectrum of $s(t)$. Delaying a signal by τ seconds results in a spectrum having a **linear phase shift** of $-\frac{2\pi k\tau}{T}$ in comparison to the spectrum of the undelayed signal. Note that the spectral magnitude is unaffected. Showing this property is easy.

PROOF

$$\frac{1}{T}\int_0^T s(t-\tau)e^{\frac{(-j)2\pi kt}{T}}\,\mathrm{d}t = \frac{1}{T}\int_{-\tau}^{T-\tau} s(t)e^{\frac{(-j)2\pi k(t+\tau)}{T}}\,\mathrm{d}t = \frac{1}{T}e^{\frac{(-j)2\pi k\tau}{T}}\int_{-\tau}^{T-\tau} s(t)e^{\frac{(-j)2\pi kt}{T}}\,\mathrm{d}t$$

(8)

Note that the range of integration extends over a period of the integrand. Consequently, it should not matter how we integrate over a period, which means that $\int_{-\tau}^{T-\tau}(\cdot)\,\mathrm{d}t=\int_0^T(\cdot)\,\mathrm{d}t$, and we have our result.

The complex Fourier series obeys **Parseval's Theorem**, one of the most important results in signal analysis. This general mathematical result says you can calculate a signal's power in either the time domain or the frequency domain.

THEOREM 1: Parseval's Theorem

Average power calculated in the time domain equals the power calculated in the frequency domain.

$$\frac{1}{T}\int_0^T s^2(t)\,\mathrm{d}t=\sum_{k=-\infty}^{\infty}\left(\left|c_k\right|\right)^2$$

(9)

This result is a (simpler) re-expression of how to calculate a signal's power than with the real-valued Fourier series expression for power.

Let's calculate the Fourier coefficients of the periodic pulse signal shown **here**.

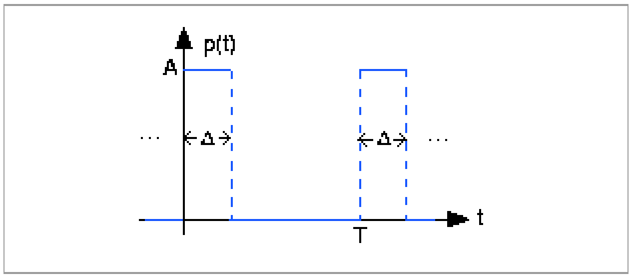

Figure 1: Periodic pulse signal.

The pulse width is Δ, the period T, and the amplitude A. The complex Fourier spectrum of this signal is given by

$$c_k = \frac{1}{T}\int_0^{\Delta} A e^{-\frac{j2\pi kt}{T}} \, dt = -\left(\frac{A}{j2\pi k}\left(e^{-\frac{j2\pi k\Delta}{T}} - 1\right)\right)$$

At this point, simplifying this expression requires knowing an interesting property.

$$1 - e^{-(j\theta)} = e^{-\frac{j\theta}{2}}\left(e^{\frac{j\theta}{2}} - e^{-\frac{j\theta}{2}}\right) = e^{-\frac{j\theta}{2}} 2j\sin\left(\frac{\theta}{2}\right)$$

Armed with this result, we can simply express the Fourier series coefficients for our pulse sequence.

$$c_k = A e^{-\frac{j\pi k\Delta}{T}} \frac{\sin\left(\frac{\pi k\Delta}{T}\right)}{\pi k}$$

(10)

Because this signal is real-valued, we find that the coefficients do indeed have conjugate symmetry: $c_k = c_{-k}^*$. The periodic pulse signal has neither even nor odd symmetry; consequently, no additional symmetry exists in the spectrum. Because the spectrum is complex valued, to plot it we need to calculate its magnitude and phase.

$$|c_k| = A\left|\frac{\sin\left(\frac{\pi k\Delta}{T}\right)}{\pi k}\right|$$

(11)

$$\angle(c_k) = -\frac{\pi k\Delta}{T} + \pi \operatorname{neg}\left(\frac{\sin\left(\frac{\pi k\Delta}{T}\right)}{\pi k}\right)\operatorname{sign}(k)$$

The function $\operatorname{neg}(\cdot)$ equals -1 if its argument is negative and zero otherwise. The somewhat complicated expression for the phase results because the sine term can be negative; magnitudes must be positive, leaving the occasional negative values to be accounted for as a phase shift of π.

Periodic Pulse Sequence

Periodic Pulse Sequence

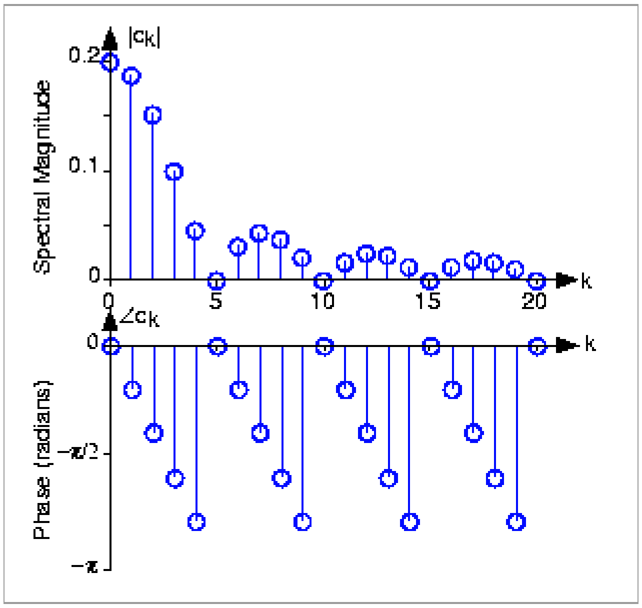

Figure 2: The magnitude and phase of the periodic pulse sequence's spectrum is shown for positive-frequency indices. Here $\frac{\Delta}{T}=0.2$ and $A=1$.

Also note the presence of a linear phase term (the first term in $\angle(c_k)$ is proportional to frequency $\frac{k}{T}$). Comparing this term with that predicted from delaying a signal, a delay of $\frac{\Delta}{2}$ is present in our signal. Advancing the signal by this amount centers the pulse about the origin, leaving an even signal, which in turn means that its spectrum is real-valued. Thus, our calculated spectrum is consistent with the properties of the Fourier spectrum.