

Advanced Stochastic Models for Partially Developed Speckle

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Abstract—Speckled images arise when coherent microwave, optical, and acoustic imaging techniques are used to image an object, surface or scene. Examples of coherent imaging systems include synthetic aperture radar, laser imaging systems, imaging sonar systems, and medical ultrasound systems. Speckle noise is a form of object or target induced noise that results when the surface of the object is Rayleigh rough compared to the wavelength of the illuminating radiation. Detection and estimation in images corrupted by speckle noise is complicated by the nature of the noise and is not as straightforward as detection and estimation in additive noise. In this work, we derive stochastic models for speckle noise, with an emphasis on speckle as it arises in medical ultrasound images. The motivation for this work is the problem of segmentation and tissue classification using ultrasound imaging. Modeling of speckle in this context involves partially developed speckle model where an underlying Poisson point process modulates a Gram-Charlier series of Laguerre weighted exponential functions, resulting in a doubly stochastic filtered Poisson point process. The statistical distribution of partially developed speckle is derived in a closed canonical form. It is observed that as the mean number of scatterers in a resolution cell is increased, the probability density function approaches an exponential distribution. This is consistent with fully developed speckle noise as demonstrated by the Central Limit theorem.

Keywords—Doubly stochastic filtered process, Poisson point process, segmentation, speckle, ultrasound.

I. INTRODUCTION

SPECKLE noise is caused by scattering of microwave and ultrasound signals from rough surfaces is undoubtedly the major obstacle in image analysis [1-6]. This includes detection, estimation, classification, and segmentation problems in coherent imaging systems [7-17].

Key to the development of efficient estimation and detection schemes in speckled images are accurate modeling and in-depth stochastic analysis of the speckle noise formation. The mapped surface structures exhibit varying degrees of roughness (on the scale of a wavelength), which induces different stochastic models for speckle noise.

Classically, speckle noise has been assumed to be fully developed, comprising an infinite number of scatterers, leading to well known models such as Rayleigh envelope or

exponential intensity, Rician envelope or modified Rician intensity, lognormal envelope or intensity, and Nakagami- m envelope or Gamma intensity. While these models may work well (but not always) for very rough surfaces, they are not consistent with speckle noise characteristics in surfaces with a finite number of scatterers. In this context, speckle is referred to as *partially developed*.

In this paper we develop stochastic models for partially-developed speckle noise which proves to be more accurate for image analysis in many applications, especially in medical ultrasound images).

II. STOCHASTIC MODELING OF PARTIALLY DEVELOPED SPECKLE NOISE

There are many applications where the mapped surface comprises a small random number of scatterers. Notable applications are soft tissues in medical ultrasound imaging and ocean surfaces with relatively small capillary waves in synthetic aperture radar (SAR) imaging.

A number of stochastic models have been presented in the literature to characterize partially developed speckle. These models include the Oliver's K distribution (simply known as the k -distribution) and the homodyned k -distribution. These models have so far acted as a "quick fix" solution to an analytically intractable exact solution to the probability density function of speckle, and at best, they serve as an approximation to the real model. Our approach addresses the shortfall in accurate modeling of partially developed speckle and is based on a doubly stochastic random point processes in space.

A. Marked Accumulator Poisson Point Process

One common approach to modeling random scattering is to assume that the backscattered return within a resolution element arises from the collection of elemental point scatterers. The total backscattered field is then taken to be the sum of the scattered fields from all of the elemental scatterers. The amplitude of the scattered field from each of these elemental scatterers will in general be a function of their size and physical properties. Thus in general, the amplitude of the field scattered by each elemental scatterer can be a random variable.

For most coherently imaged surfaces, the locations of the elemental scatterers can be viewed as random, and furthermore, the number of elemental scatters within a resolution cell will be a random variable. One of the most effective methods for modeling "elements" that occur

randomly in space is to use a *point process* [18]. If, in addition, there is attached to each point (random location) a random quantity that can be represented by one or more random variables (in this case the amplitude and phase of the backscattered field), the natural stochastic model to use is that of a *marked point process* [18]. Furthermore, for many surfaces, the number of elemental scatterers in disjoint regions will be statistically independent integer-valued random variables and the point process is termed a *compound point process* [18]. If in addition, the point process satisfies a technical condition called *Khinchine orderliness* [18], then the point process will be a Poisson point process. The number of elemental scatterers within a resolution cell will be a Poisson random variable with intensity or rate λ , which in turn is random. For this reason, a *doubly stochastic compound Poisson point process* is a useful model of random scattering in speckled images.

When the number of scattering points within a resolution cell is sufficiently large that the central limit theorem holds and the scattered field is approximately a circular complex Gaussian random variable, we say that the speckle noise is *fully developed*. But if the number of points is relatively small typically less than 10 to 20, the speckle noise is not fully developed. In this case, the doubly stochastic compound point process is useful for characterizing the *partially developed speckle*, and estimates of the intensity λ can be used to characterize the average intensity of a resolution cell. We now investigate the use of doubly stochastic compound point process for this purpose.

B. Physical Model of a Resolution Cell

The fading power ν_N obeys the mathematical model

$$\nu_N = \left| \sum_{k=1}^N A_k e^{j\phi_k} \right|^2 \quad (1)$$

whose parameters are defined as follows:

A_k - The reflectance strength of the k -th scatterer, a measure of energy absorption of the scattered wavelets by the k -scatterer.

In general, A_k are random.

ϕ_k - The phase of the k -th scatterer, a measure of the delay of scattered wavelets caused by path deflection. $\phi_k \sim U[-\pi, \pi)$.

N - The number of scatterers in an area A . $N \sim \text{Poisson}(\lambda A)$, the Poisson intensity λ is the density of the scatterers per unit area.

C. Fully Developed Speckle

The complex echo $Z_N = \sum_{k=1}^N A_k e^{j\phi_k}$ may be grouped into

two components: In-phase component

$$Z_{X,N} = \sum_{k=1}^N A_k \cos(\phi_k) \quad \text{and} \quad \text{quadrature component}$$

$$Z_{Y,N} = \sum_{k=1}^N A_k \sin(\phi_k), \text{ such that } Z_N = Z_{X,N} + jZ_{Y,N}.$$

Provided that all the individual amplitudes $\{a_k\}$ remain small relative to the total power, $Z_{X,N}$ and $Z_{Y,N}$ will be uncorrelated. The central limit theorem implies that $Z = \lim_{N \rightarrow \infty} Z_N$ is asymptotically circularly zero-mean

Gaussian. Even if the scatterers were statistically dependent, $Z = \lim_{N \rightarrow \infty} Z_N$ would still be asymptotically Gaussian hinging on the fact that the sequence of scattering random variables satisfies the α -mixing property or the conditions of the Lindeberg-Feller form of the central limit theorem [19, sec. 27].

Under these assumptions, the delay-spread function is modeled as a zero-mean complex-valued Gaussian process. The envelope $\gamma = |Z|$ is thus Rayleigh distributed and the intensity (power) $\nu = \gamma^2$ is exponentially distributed (chi-square with 2 degrees of freedom). This situation corresponds to *fully developed speckle* when a very large number of scatterers are present in the imaged resolution cell.

D. Exact Probability Density Function

In this section, we present exact expressions for the probability density function (pdf) of partially developed speckle intensity with fixed reflectances and finite deterministic number of scatterers.

The conditional speckle intensity $\nu_N | N$ for the generalized random walk in the complex plane model

$$\nu_N = \left| \sum_{k=1}^N A_k e^{j\phi_k} + V_0 \right|^2 \quad (2)$$

can be generated from the definite integral [Abr70, Gra94]

$$p_{\nu_N}(\nu | N) = \frac{1}{2} \int_0^\infty u J_0(u\sqrt{\nu}) e^{-\frac{u^2 |V_0|^2}{4}} \left[\prod_{k=1}^N J_0(a_k u) \right] du, \quad (3)$$

where $|V_0|^2$ is the intensity of a present coherent background, and $J_0(\cdot)$ is the zero-order Bessel function of the first kind. By

setting $g(u) = e^{-\frac{u^2 P_c}{4}} \prod_{k=1}^N J_0(a_k u)$, $p_{\nu_N}(\nu)$ can be conceived

as $0.5 H_{0,\sqrt{\nu_N}}\{g(u)\}$, where $H_{0,z}\{g(u)\}$ is the zero-th order Hankel transform (or Fourier-Bessel transform) of $g(u)$ defined as

$$H_{0,z}\{g(u)\} = \int_0^\infty u J_0(zu) g(u) du. \quad (4)$$

Different model parameters $(a_k, N, |V_0|^2)$ will generate different functions $g(\cdot)$ whose Hankel transforms can be obtained, when feasible, from the Hankel tables published in [Oberhettinger72].

E. Doubly Stochastic Filtered Poisson Point Process

Without loss of generality, let us assume $A_k = A_0, \forall k$, and $V_0 = 0$. The pdf of the speckle intensity ν is obtained by simply averaging the conditional pdf of Eq.(3) over a Poisson distribution (assuming independence between N and ν_N):

$$p_\nu(\nu) = E_N \left(p_{\nu_N|N}(\nu | N = n) \right) \\ = e^{-\lambda A} \sum_{n=1}^{\infty} \frac{(\lambda A)^n}{n!} p_{\nu|N}(\nu | N = n). \quad (5)$$

In a separate paper [7], a closed form for the conditional speckle intensity $\nu_N | N$ is developed for Eq.(3) in terms of a Gram-Charlier series expansion. Based on these results, we further develop the speckle intensity ν_N and show that it obeys a series of Laguerre-weighted exponential law with parameters $\lambda A > 0$ and $A_0 > 0$. This is denoted by $\nu \sim$ Laguerre-weighted exponential mixture($\lambda A, A_0$). The canonical form of Eq. (5) is presented below for different cases.

Case I: $0 < \nu < A_0^2$

$$p_\nu(\nu) = e^{-\lambda A} \left(\lambda A \delta(\nu - A_0^2) + \frac{(\lambda A)^2}{2\pi \sqrt{\nu(4A_0^2 - \nu)}} \right. \\ \left. + \frac{(\lambda A)^3}{3\pi^2 \sqrt{(\sqrt{\nu} + A_0)^3 (3A_0 - \sqrt{\nu})}} K \left(\frac{4A_0^{3/2} \nu^{1/4}}{\sqrt{(\sqrt{\nu} + A_0)^3 (3A_0 - \sqrt{\nu})}} \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=4}^8 \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \left(1 + \sum_{m=1}^{M_n} c_m L_m \left(\frac{\nu}{nA_0^2} \right) \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=9}^{\infty} \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \right), \quad (6)$$

where $L_m(x)$ are Laguerre polynomials defined by their expansion in powers of x

$$L_m(x) = \sum_{j=0}^m (-1)^j \binom{m}{m-j} \frac{x^j}{j!}. \quad (7)$$

The pdf $p_\nu(\nu)$ has a singularity at $\nu = A_0^2$.

The coefficients c_m measure the departure of the conditional pdf $p_{\nu|N}(\nu|N=n)$ from a pure exponential law and are estimated in Table I using a maximum likelihood estimator equal to the sample mean of a random sample of diffuse power $\{\nu_k\}$

$$\hat{c}_m = \frac{1}{K} \sum_{k=1}^K L_m \left(\frac{\nu_k}{nA_0^2} \right). \quad (8)$$

The power measurements $\{\nu_k\}$, $k = 1, 2, \dots, K$, are generated using computerized Monte-Carlo simulation. We

note that exact values of c_m can be obtained according to the work published by [20].

Case II: $A_0^2 < \nu < 4A_0^2$

$$p_\nu(\nu) = e^{-\lambda A} \left(\lambda A \delta(\nu - A_0^2) + \frac{(\lambda A)^2}{2\pi \sqrt{\nu(4A_0^2 - \nu)}} \right. \\ \left. + \frac{(\lambda A)^3}{12\pi^2 A_0^{3/2} \nu^{1/4}} K \left(\frac{\sqrt{(\sqrt{\nu} + A_0)^3 (3A_0 - \sqrt{\nu})}}{4A_0^{3/2} \nu^{1/4}} \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=4}^8 \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \left(1 + \sum_{m=1}^{M_n} c_m L_m \left(\frac{\nu}{nA_0^2} \right) \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=9}^{\infty} \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \right), \quad (9)$$

where $K(\cdot)$ is the elliptic-K function. The pdf $p_\nu(\nu)$ has singularities at $\nu = A_0^2, 4A_0^2$.

Case III: $4A_0^2 < \nu \leq 9A_0^2$

$$p_\nu(\nu) = e^{-\lambda A} \left(\frac{(\lambda A)^3}{12\pi^2 A_0^{3/2} \nu^{1/4}} K \left(\frac{\sqrt{(\sqrt{\nu} + A_0)^3 (3A_0 - \sqrt{\nu})}}{4A_0^{3/2} \nu^{1/4}} \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=4}^8 \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \left(1 + \sum_{m=1}^{M_n} c_m L_m \left(\frac{\nu}{nA_0^2} \right) \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=9}^{\infty} \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \right). \quad (10)$$

The pdf $p_\nu(\nu)$ has a singularity at $\nu = 4A_0^2$.

Case IV: $\nu \geq 9A_0^2$

$$p_\nu(\nu) = e^{-\lambda A} \left(\frac{1}{A_0^2} \sum_{n=4}^8 \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \left(1 + \sum_{m=1}^{M_n} c_m L_m \left(\frac{\nu}{nA_0^2} \right) \right) \right. \\ \left. + \frac{1}{A_0^2} \sum_{n=9}^{\infty} \frac{(\lambda A)^n}{n!n} e^{-\frac{\nu}{nA_0^2}} \right). \quad (11)$$

III. SIMULATIONS AND RESULTS

Without loss of generality and for the purpose of simulation, we assume that $A_0 = 1$. Fig. 1 illustrates the pdf of the partially developed speckle intensity for different mean number of scatterers λA . We observe that the shape of the pdf is a concatenation of the expressions of the conditional

pdfs in the 4 cases (Eq.(6), Eq.(9), Eq.(10), and Eq.(11)). We also observe that as the mean number of scatterers increase from 2 to 8, the pdf converges to an exponential distribution as dictated by the Central Limit theorem.

TABLE I
ESTIMATED LAGUERRE POLYNOMIALS WEIGHTING COEFFICIENTS c_m

m	No. scatterers n				
	4	5	6	7	8
1	1.07203E-4	4.41722E-4	-1.59098E-4	-3.00960E-4	8.0130E-04
2	-1.24886E-1	-1.00232E-1	-8.34411E-2	-7.11855E-2	
3	-4.20834E-2	-2.65807E-2	-1.82479E-2		
4	2.50561E-2	1.95443E-2			
5	3.74666E-2	2.13131E-2			
6	1.59254E-2	4.44057E-3			
7	-1.09051E-2	-8.35288E-3			
8	-2.59725E-2				
9	-2.34329E-2				
10	-7.92700E-3				
11	1.36916E-2				
12	3.46515E-2				
13	4.95530E-2				
14	5.58110E-2				
15	5.42184E-2				
16	4.56010E-2				
17	3.30694E-2				

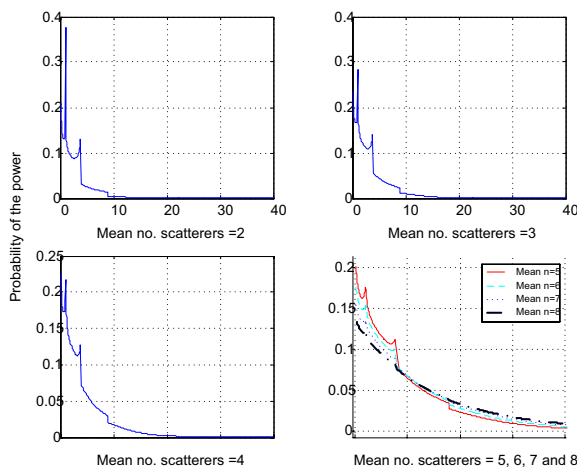


Fig. 1 PDF of partially developed intensity for single look speckle with random number of scatterers

IV. CONCLUSION AND FUTURE WORKS

In this paper, we presented canonical forms for the statistical distribution of partially developed speckle intensity. The models were derived based on previously developed Gram-Charlier type series of Laguerre weighted exponential functions for the conditional pdf of speckle intensity. An underlying Poisson point process modulates the speckle conditional pdf, yielding a doubly stochastic filtered Poisson point model. This new model was simulated for different parameters in terms of mean number of scatterers. The results were consistent with the Central Limit Theory in the sense that the distribution asymptotically converged to an exponential law with an increasing mean number of scatterers. These results form a basis for estimation, detection, segmentation, and general image analysis in speckle noise and have an impact on applications such as medical ultrasound and synthetic aperture radar systems.

As future work, we plan on developing estimation algorithms for the intensity λ of the Poisson point process.

More cases of surface characterizations can also be examined where a number of scattering *centers* (as opposed to points) is present. The implications of such characterization would be that the variable A_k becomes random under Rayleigh or Rician laws. This complicates the statistics even further since the process becomes *triple stochastic*.

Another alternative model for study is a doubly stochastic filtered point process driven by a Pascal-Negative Binomial model for the number of scatterers [21 - 23].

We also plan on deriving stochastic models for multi-look partially developed speckle.

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