Optimization of Plate Fin Heat Sinks Using Entropy Generation Minimization

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Abstract—The specification and design of heat sinks for electronic applications is not easily accomplished through the use of conventional thermal analysis tools because "optimized" geometric and boundary conditions are not known a priori. A procedure is presented that allows the simultaneous optimization of heat sink design parameters based on a minimization of the entropy generation associated with heat transfer and fluid friction. All relevant design parameters for plate fin heat sinks, including geometric parameters, heat dissipation, material properties and flow conditions can be simultaneously optimized to characterize a heat sink that minimizes entropy generation and in turn results in a minimum operating temperature. In addition, a novel approach for incorporating forced convection through the specification of a fan curve is integrated into the optimization procedure, providing a link between optimized design parameters and the system operating point. Examples are presented that demonstrate the robust nature of the model for conditions typically found in electronic applications. The model is shown to converge to a unique solution that gives the optimized design conditions for the imposed problem constraints.

Index Terms—Electronics cooling, entropy production, forced convection, heat sink, optimization.

NOMENCLATURE

A	Surface area in m^2.
A_c	Cross-sectional area in m^2.
A_f	Total frontal area of heat sink $\equiv H \cdot W, m^2$.
A_p	Fin profile area $\equiv N \cdot H \cdot t, m^2$.
b	Fin spacing in m.
D_h	Hydraulic diameter m $\equiv 2b,$.
f, f_{app}	Friction factor and apparent friction factor, respectively.
F_d	Drag force $kg \cdot m/s^2$.
$g_i()$	Minimizing function as in (21).
h	Heat transfer coefficient W/m^2 K.
H	Fin height in m.
k	Thermal conductivity in W/mK.
K_c, K_e	Contraction and expansion loss coefficient, respectively.
L	Fin length in m.
L^*	Dimensionless fin length.
m	Fin parameter $\approx \sqrt{hP/kA_c}$, m^{-1}.
N	Number of fins.
P	Perimeter in m.
ΔP	Pressure drop in mm_{H_2O}.
Pr	Prandtl number.
Q	Heat flow rate in W.
R	Thermal resistance in °C/W.
Re_{D_h}	Channel Reynolds number $\equiv D_h \cdot V_{ch}/\nu$.
Re^*	Modified channel Reynolds number $\equiv (b \cdot V_{ch}/\nu) \cdot (b/L)$.
$\dot{S}_{gen}$	Entropy generation rate in W/°C.
t	Fin thickness in m.
t_b	Base plate thicknees in m.
T	Temperature in °C.
V	Velocity in m/s.
V_{ch}	Channel velocity in m/s.
W	Heat sink width in m.

Greek

ρ	Density in kg/m^3.
σ	Fraction of frontal free flow area.
θ_b	Temperature excess $\equiv (T_b - T_0), K$.
$\dot{\ominus}$	Volumetric flow rate in l/s.

Subscripts

0	Ambient.
b	Baseplate.
ch	Channel.
f	Film.
fin	Fin.
f	Fluid.
sink	Heat sink.

I. INTRODUCTION

THE ability of a designer to minimize the thermal resistance between the source of heat dissipation and the thermal sink is essential in controlling maximum operating temperatures and consequently the long term reliability and performance of electronic components. Typical electronic packages can introduce a complex network of resistive paths as heat passes from the integrated circuit through various laminated structures, bonding adhesives, lead frames or sometimes ball grid arrays. Despite the multitude of materials and interfaces within an electronic

package, the largest thermal resistance, and consequently the controlling resistance in the path between the source and the sink, is usually the boundary layer or film resistance. Given the relationship in (1), an increase in either the heat transfer coefficient or the surface area for heat transfer results in a reduction in the film resistance.

$$R_{\mathrm{f}} = \frac{1}{h \cdot A}. \tag{1}$$

While the convective heat transfer coefficient could potentially be enhanced with an increase in the approach velocity, the dependence of the heat transfer coefficient on the square root of the velocity in laminar flow results in diminished returns as velocity is increased. In addition, noise constraints associated with many electronics applications restrict flow velocities to a range of 5 m/s or less. The second option for reducing film resistance is achieved by increasing the effective surface area for convective heat transfer. This is typically achieved through the use of heat sinks and extended surfaces.

Heat sinks offer a low cost, convenient method for lowering the film resistance and in turn maintaining junction operating temperatures at a safe level for long term, reliable operation. Unfortunately, the selection of the most appropriate heat sink for a particular application can be very difficult given the many design options available. Thermal analysis tools, ranging from simple empirically derived correlations to powerful numerical simulation tools, can be used to analyze the thermal performance of heat sinks for a given set of design conditions. Regardless of which procedure is used, analysis tools only provide a performance assessment for a prescribed design where all design conditions are specified *a priori*. Following an exhaustive parametric analysis, design options can be assessed with respect to their influence on thermal performance, however, there is no guarantee that an "optimized" solution is obtained since the parametric analysis only provides a ranking of a limited set of test cases. The method of entropy generation minimization, pioneered by Bejan [1]–[4], provides a procedure for simultaneously assessing the parametric relevance of system parameters as they relate to not only thermal performance but also viscous effects.

The following procedures provide a detailed application of Bejan's approach for plate fin heat sinks commonly found in electronic cooling networks. The solution procedure allows for single parameter optimization [3], where any design parameter can be optimized while all other design conditions are set. In addition a procedure is presented for multi-parameter optimization where any number of parameters can be simultaneously optimized, providing the optimum design for the given conditions.

While some situations exist where the approach velocity at the inlet of a plate fin heat sink is known, a far more common scenario is the use of an axial or muffin fan where the velocity is regulated based on the pressure drop across the heat sink. The optimization approach presented in the following section establishes a direct link between the pressure drop of the optimized heat sink and the system operating curve for the selected fan.

Examples will be presented for a variety of conditions typically observed in electronics applications.

II. Model Development

Numerous analysis tools are available for determining the thermal performance of heat sinks given a well defined set of design conditions. Convective optimizations are available, such as those presented in Kraus and Bar-Cohen [5], however, these models assume a prescribed heat transfer coefficient over the length of the fins which is constant, while in most heat sink applications, hydrodynamic, and thermal entrance effects introduce a variable heat transfer coefficient, at least over a portion of the heat sink. The assumption of a constant value of heat transfer coefficient can no longer be prescribed, since the value will depend upon fin spacing and length in the direction of flow. Optimization routines that lead to changes in fin spacing, fin height or fin length also result in changes in the mean heat transfer coefficient and head loss in such a way that iterative procedures are required. While in some instances parametric studies can be undertaken to obtain a relationship between thermal performance and design parameters, a comprehensive design tool should also take into consideration the effect of viscous dissipation and its relationship on thermal performance. The entropy generation associated with heat transfer and frictional effects serve as a direct measure of lost potential for work or in the case of a heat sink, the ability to transfer heat to the surrounding cooling medium. A model that establishes a relationship between entropy generation and heat sink design parameters can be optimized in such a manner that all relevant design conditions combine to produce the best possible heat sink for the given constraints.

Assumptions: The model development is subject to the following underlying assumptions:

1) no spreading or constriction resistance;
2) no contact resistance at fin to base plate connection;
3) no bypassing of flow;
4) uniform approach velocity;
5) constant thermal properties;
6) uniform heat transfer coefficient;
7) adiabatic fin tips.

Heat Sink Model: The entropy generation rate for extended surfaces in external flow with conductive resistance is defined by the following relationship [1], [3]

$$\dot{S}_{\mathrm{gen}} = \frac{Q\theta_{\mathrm{b}}}{T_{\mathrm{o}}^2} + \frac{F_{\mathrm{d}}V_{\mathrm{f}}}{T_{\mathrm{o}}} \tag{2}$$

where
Q heat dissipation rate;
θ_{b} temperature excess of the heat sink base plate;
F_{d} total drag force;
V_{f} free stream or approach velocity;
T_{o} absolute environment temperature.

The temperature excess of the heat sink θ_{b} may be related to the overall heat sink resistance by

$$\theta_{\mathrm{b}} = Q \cdot R_{\mathrm{sink}} \tag{3}$$

such that

$$\dot{S}_{\mathrm{gen}} = \frac{Q^2 R_{\mathrm{sink}}}{T_{\mathrm{o}}^2} + \frac{F_{\mathrm{d}}V_{\mathrm{f}}}{T_{\mathrm{o}}}. \tag{4}$$

Entropy generation is clearly a function of both heat sink resistance and head loss. Under some flow conditions, such as low

velocity, buoyancy induced flows, the fluid friction component of the entropy generation is small and can be neglected. However, an a priori assumption with respect to the contribution of the terms in (4), significantly restricts the use of the general optimization model. The inclusion of the viscous component is essential for "optimal" flow conditions to be determined over the wide range of flow conditions encountered in air cooled heat sinks.

The overall heat sink resistance is given by

$$R_{\text{sink}} = \frac{1}{(N/R_{\text{fin}}) + h(N-1)bL} + \frac{t_{\text{b}}}{kLW} \quad (5)$$

where N is the number of fins, and R_{fin} is the thermal resistance of a single fin. It will be modeled using the solution for a straight fin with an adiabatic tip

$$R_{\text{fin}} = \frac{1}{\sqrt{hPkA_{\text{c}}}\tanh(mH)} \quad (6)$$

where

$$m = \sqrt{\frac{hP}{kA_{\text{c}}}} \quad (7)$$

and P is the perimeter of the fin and A_{c} is the cross-sectional area of the fin.

The total drag force on the heat sink may be obtained by considering a force balance on the heat sink, Kays and London [6]

$$\frac{F_{\text{d}}}{(1/2)\rho V_{\text{ch}}^2} = f_{\text{app}}N(2HL+bL)+K_{\text{c}}(HW)+K_{\text{e}}(HW) \quad (8)$$

where f_{app} is the apparent friction factor for hydrodynamically developing flow, and the channel velocity, V_{ch}, is related to the free stream velocity by

$$V_{\text{ch}} = V_{\text{f}}\left(1+\frac{t}{b}\right). \quad (9)$$

1) Channel Models: The apparent friction factor, f_{app}, for a rectangular channel may be computed using a form of the model developed by Muzychka and Yovanovich [7] for developing laminar flow

$$f_{\text{app}}\text{Re}_{D_{\text{h}}} = \left[\left(\frac{3.44}{\sqrt{L^*}}\right)^2 + (f\text{Re}_{D_{\text{h}}})^2\right]^{1/2} \quad (10)$$

where

$$L^* = \frac{L}{D_{\text{h}}\text{Re}_{D_{\text{h}}}} \quad (11)$$

and D_{h} is the hydraulic diameter of the channel and $f \cdot \text{Re}_{D_{\text{h}}}$ is the fully developed flow friction factor Reynolds number group given by

$$\begin{aligned} f \cdot \text{Re}_{D_{\text{h}}} =& 24 - 32.527\left(\frac{b}{H}\right) + 46.721\left(\frac{b}{H}\right)^2 \\ &- 40.829\left(\frac{b}{H}\right)^3 + 22.954\left(\frac{b}{H}\right)^4 \\ &- 6.089\left(\frac{b}{H}\right)^5. \end{aligned} \quad (12)$$

The expansion and contraction loss coefficients may be computed using the simple expressions for a sudden contraction and a sudden expansion [8].

$$K_{\text{c}} = 0.42(1-\sigma^2) \quad (13)$$
$$K_{\text{e}} = (1-\sigma^2)^2 \quad (14)$$

where

$$\sigma = 1 - \frac{N \cdot t}{W.} \quad (15)$$

The heat transfer coefficient, h, will be computed using the model developed by Teertstra *et al.* [9].

$$Nu_{\text{b}} = \left[\left(\frac{\text{Re}_{\text{b}}^*\text{Pr}}{2}\right)^{-3} + \left(0.664\sqrt{\text{Re}_{\text{b}}^*}\text{Pr}^{1/3}\sqrt{1+\frac{3.65}{\sqrt{\text{Re}_{\text{b}}^*}}}\right)^{-3}\right]^{-1/3} \quad (16)$$

where

$$Nu_{\text{b}} = \frac{h \cdot b}{k_{\text{f}}} \quad (17)$$

$$Re_{\text{b}}^* = Re_{\text{b}} \cdot \left(\frac{b}{L}\right). \quad (18)$$

These models have been validated against experimental and numerical data (Muzychka and Yovanovich [7], Teertstra *et al.* [9]), and found to predict the performance of heat sink data quite well.

III. PARAMETRIC OPTIMIZATION

The rate of entropy generation given in (4) can be used to optimize any or all parameters in a plate fin heat sink. The simplest approach to entropy generation minimization is obtained by fixing all parameters in the heat sink design but one and then monitoring the change in entropy generation as that particular design variable is changed over a typical range. A distinct minimum will be established that represents the magnitude of the free variable that leads to the lowest rate of entropy generation. The same result can be achieved by solving for the free variable, x, in the following relationship

$$\frac{\partial \dot{S}_{\text{gen}}}{\partial x} = 0. \quad (19)$$

While single parameter optimization can provide an optimized design condition when all other design parameters are predetermined, there is no guarantee that this "optimized" result will hold when other design parameters are unconstrained. Optimization must be achieved based on a simultaneous solution considering all unconstrained variables. This can be accomplished by incorporating a multi-parameter Newton–Raphson method where the minimizing equation given in (19) is invoked for each unconstrained variable, leading to a series of nonlinear equations that must be solved in a simultaneous manner. The approach used to solve the system of equations is summarized in Stoecker [10] and is presented as follows:

$$\dot{S}_{\text{gen}} = f(x_1, x_2, x_3, \dots x_N) \quad (20)$$
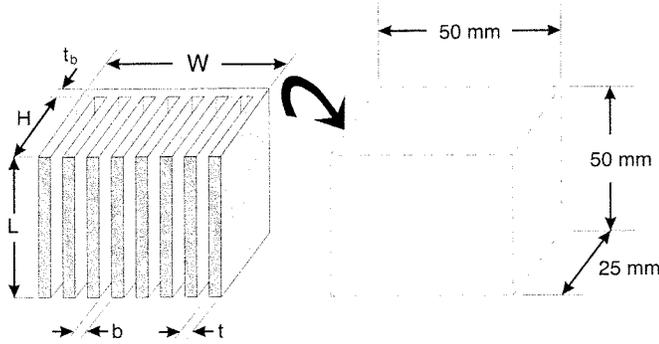
Fig. 1. Heat sink design parameters.

and

$$\frac{\partial \dot{S}_{\text{gen}}}{\partial x_i} = g_i = 0 \quad \text{for } i = 1, 2, 3, \ldots N \qquad (21)$$

where the x_i's are the unconstrained variables in the problem of interest.

A series of nonlinear equations is obtained that can be solved using a Newton–Raphson Method for multiple equations and unknowns, as follows:

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_N} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_N} \\ & & \vdots & \\ \frac{\partial g_N}{\partial x_1} & \frac{\partial g_N}{\partial x_2} & \cdots & \frac{\partial g_N}{\partial x_N} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_N \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}. \qquad (22)$$

The solution procedure is initiated by assuming an initial value for the unconstrained variables that leads to an improved estimate once the equations are solved. The difference between successive values of the unconstrained variables, as given by δx_i, is forced to zero (less than a predetermined tolerance) such that

$$g_i(\text{guess}) \approx g_i(\text{actual}) + g_i'(\text{guess}) \cdot \delta x_i. \qquad (23)$$

The method usually converges after a few iterations provided a good initial guess is made. Good design intuition should provide an adequate initial guess. More robust methods are available for solving nonlinear equations or systems of equations, however, this simple procedure is easily adapted in most mathematical software packages.

IV. DISCUSSION

The entropy generation minimization approach will be demonstrated by applying the methods described above to a set of typical design parameters found in electronic applications. It is not unusual for a designer to be given an overall maximum heat sink volume that is determined by the foot print of the electronic package being cooled and the board-to-board spacing between adjacent printed wiring boards. The examples presented below are assumed to be constrained by a overall maximum volume of 50 mm $\times$ 50 mm $\times$ 25 mm, as shown in Fig. 1.

It is also assumed that a total heat dissipation of 30 W is uniformly applied over the base plate of the heat sink which has

TABLE I
SINGLE PARAMETER OPTIMIZATION—SOLVE FOR N

iter.	N	θ_b ($^\circ C$)	ΔP ($mm_{H_2 0}$)	Re
1	10.00	36.20	0.46	1451.9
2	15.14	21.75	0.94	923.9
3	21.56	14.30	2.16	635.4
4	26.74	11.94	4.34	507.6
5	28.51	11.52	5.57	475.0
6	28.57	11.51	5.62	473.9
7	28.57	11.51	5.62	473.9

a uniform thickness of 2 mm. Other constraints that are fixed are the thermal conductivity of the heat sink at $k = 200$ W/mK and the ambient temperature of the surrounding air medium at $T_o = 25^\circ$C or 298 K.

Six cases are presented that demonstrate the method of entropy generation minimization for sizing plate fin heat sinks. The examples include single and multiparameter optimization as well as a unique approach for optimizing heat sinks using a fan curve where the system operating point is determined based on the results of the optimization procedure. Cases (i) to (v) demonstrate the effect of introducing progressively more unconstrained variables into the optimization procedure. Case (vi) is a single parameter optimization where the forced flow through the heat sinks is driven by a muffin fan, specified using a fan curve.

The system of nonlinear equations for each of the following cases can be solved using numerical procedures contained within commercially available algebraic software tools, such as Maple V [11].

Case (i)—Solve for N: Given the geometric constraints shown in Fig. 1 and a uniform heat load to the base plate of the heat sink of 30 W, an optimum number of fins, N, is to be determined when $V_f = 2$ m/s, $t = 1$ mm, and $H = 25$ mm. As shown in Table I, the estimation of the appropriate number of fins to satisfy the balance between heat transfer and viscous effects, converges in six to seven iterations to a value of $N \approx 29$.

It is easily seen that decreasing the number of fins leads to an increase in the thermal resistance of the heat sink which in turn leads to an increase in the temperature excess and a resultant increase in the entropy generation rate.

Increasing the number of fins beyond the optimized value would lead to a decrease in the heat sink resistance and temperature excess, but the increase in the head loss associated with fluid drag results in an increase in the entropy generation rate.

While the optimization procedure estimates the optimum number of fins to be 28.57 the relatively wide range of near minimum entropy generation rate between $20 < N < 35$, as shown in Fig. 2, provides designers with a range of options when specifying the appropriate number of fins. In subsequent applications of the optimization method, additional design variables are introduced into the procedure to simultaneously consider multiple parameters that lead to an optimization of the temperature excess and the head loss of the heat sink.

Fig. 3 shows an inverse relationship between heat sink resistance and pressure drop with respect to the number of fins. A heat sink optimization procedure must include the simultaneous
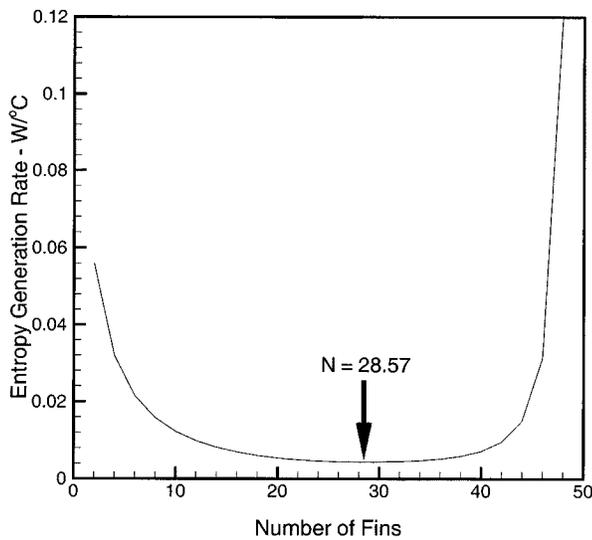
Fig. 2. Entropy generation rate versus number of fins.
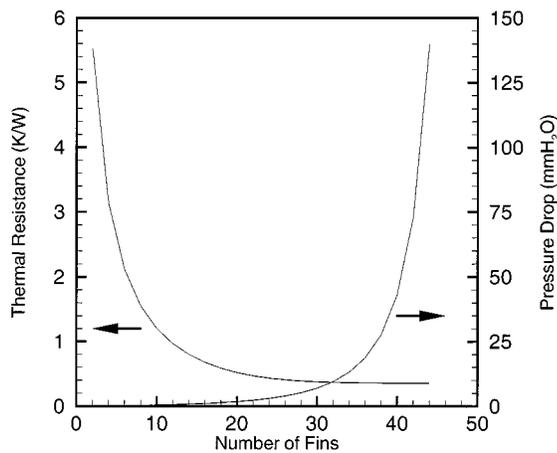
Fig. 3. Contribution of heat sink resistance and pressure drop as a function of the number of fins.

TABLE II
TWO PARAMETER OPTIMIZATION—SOLVE FOR N AND V_f

iter.	N	V_f (m/s)	θ_b (°C)	ΔP (mm_{H_2O})	Re
1	25.00	3.00	9.79	6.10	816.7
2	26.76	2.80	9.52	6.96	710.2
3	26.77	2.81	9.49	7.02	713.3
4	26.77	2.81	9.49	7.02	713.3

TABLE III
THREE PARAMETER OPTIMIZATION—SOLVE FOR N, V_f, AND t

iter.	N	V_f (m/s)	t (mm)	θ_b (°C)	ΔP (mm_{H_2O})	Re
1	25.00	2.00	10.0	12.51	3.41	544.4
2	29.89	2.64	7.0	9.96	5.00	601.3
3	37.34	3.22	3.0	9.63	4.23	586.6
4	39.42	3.35	0.4	8.58	5.87	577.2
5	38.20	3.28	0.4	8.67	5.75	583.5
6	38.14	3.28	0.4	8.66	5.78	583.9
7	38.14	3.28	0.4	8.66	5.78	583.9

TABLE IV
THREE PARAMETER OPTIMIZATION—SOLVE FOR N, V_f, AND H

iter.	N	V_f (m/s)	H (mm)	θ_b (°C)	ΔP (mm_{H_2O})	Re
1	25.00	2.00	25.0	12.51	3.41	544.4
2	26.42	2.27	33.6	8.80	4.92	582.2
3	25.04	1.97	48.2	7.94	3.36	536.1
4	24.64	1.74	64.7	7.47	2.69	481.9
5	24.57	1.59	80.6	7.28	2.34	439.9
6	24.57	1.51	91.7	7.22	2.19	417.6
7	24.57	1.48	96.0	7.21	2.14	410.6
8	24.57	1.48	96.6	7.21	2.13	409.8
9	24.57	1.48	96.6	7.21	2.13	409.8

interaction of both heat sink resistance and viscous dissipation in order to ascertain optimal operating conditions.

Case (ii)—Solve for N and V_f: Case (ii) examines the effect of relaxing the constraint on free stream velocity prescribed in Case (i) while all other assumed constraints remain unchanged. As shown in Table II, the optimized number of fins is determined to be $N \approx 27$ and the approach velocity is estimated to be $V_f = 2.81$ m/s for minimum entropy generation. A decrease in the number of fins and an increase in the free stream velocity lead to a heat sink with a lower temperature excess but a higher head loss. Overall, the entropy generation rate for this case is lower than in the previous example.

Case (iii)—Solve for N, V_f, and t: The next case examines a three parameter optimization where the constraint on the fin thickness (1 mm in the previous two examples) is removed. The optimization will simultaneously balance heat transfer, both internal and external, and viscous effects such that optimized values for the free stream velocity V_f, the number of fins N, and the fin thickness t are obtained. The results of the optimization give $N \approx 38$, $V_f = 3.28$ m/s, and $t = 0.4$ mm after seven iterations, as shown in Table III. Further gains have been made in

lowering the heat sink temperature excess and head loss which result in yet a further decrease in the entropy generation rate. However, the fin thickness may be too thin for practical manufacturing considerations.

Case (iv)—Solve for N, V_f, and H: Case (iii) examined the effect of fin thickness on entropy generation rate. Many heat sink extrusion processes are limited to a single fin thickness or at least to a range of fin thicknesses that allow for ease of manufacture. It may be a more logical choice to set the fin thickness and optimize the fin height. A common rule of thumb for extrusion processes is to limit the fin height to fin thickness ratio to 10:1, however, aspect ratios greater than this can be achieved using a fabricated fin process where the fins are mechanically bonded to the base plate. The optimization procedure will assume a fin thickness of 1 mm and the fin height will be unconstrained. The results of the optimization give $N \approx 25$, $V_f = 1.48$ m/s, and $H = 96.6$ mm after nine iterations, as shown in Table IV. Although, the heat sink is now much larger than in previous cases, further gains in lowering the heat sink temperature excess and head loss have been made. This in turn will lead to a lower entropy generation rate. Unfortunately, the heat sink dimensions exceed the constraint of overall height prescribed in Fig. 1.

Case (v)—Solve for N, V_f, t, and H: Finally, none of the variables of interest will be constrained to pre-determined values, thus providing a simultaneous optimization of all design

TABLE V
FOUR PARAMETER OPTIMIZATION—SOLVE FOR N, V_f, t, AND H

iter.	N	V_f (m/s)	t (mm)	H (mm)	θ_b (°C)	ΔP (mm_{H_2O})	Re
1	25.00	2.00	10.0	25.0	12.51	3.41	544.4
2	27.82	2.31	8.0	33.0	8.98	4.03	565.1
3	28.91	2.21	0.7	45.0	7.86	3.37	520.3
4	25.66	1.75	0.9	63.0	7.63	2.56	465.0
5	23.47	1.55	1.1	81.0	7.31	2.33	449.5
6	21.07	1.35	1.3	99.0	7.30	1.99	437.0
7	20.03	1.27	1.4	113.0	7.21	1.96	433.0
8	19.22	1.22	1.5	120.0	7.22	1.89	433.1
9	19.09	1.21	1.6	122.0	7.20	1.90	433.2
10	19.07	1.21	1.6	122.0	7.20	1.90	433.3
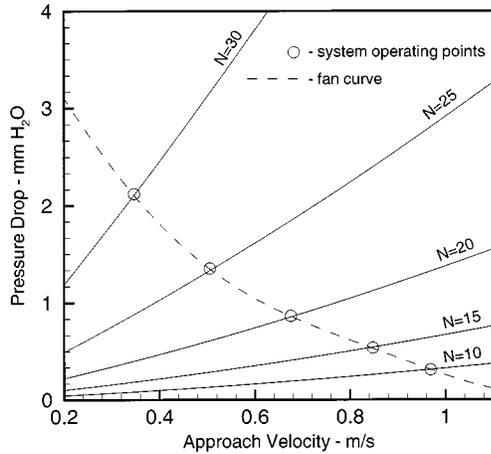
Fig. 4. System operating points for various number of fins.

variables, including the free stream velocity V_f, the number of fins N, the fin thickness t, and the fin height H. The results of the optimization give $N \approx 19$, $V_f = 1.21$ m/s, $t = 1.6$ mm, and $H = 122$ mm after ten iterations, as shown in Table V. Once again a more optimal solution has been found. While the approach presented provides an optimized heat sink, the fin height exceeds the maximum allowable height of 25 mm predicated by the board-to-board spacing, as shown in Fig. 1. Future modifications of the optimization procedure will include the ability to lock in constraints on any variable, including fin height and maximum allowable temperature.

Case (vi)—Solve for N using Fan Curve: The single parameter optimization examined in Case (i) is re-examined with an alternate approach for specifying the free stream velocity. In Case (i) the free stream velocity was specified at 2 m/s, independent of the head loss produced by the heat sink. As more fins are introduced in the optimization procedure, the velocity is maintained at 2 m/s. While this is easily achieved in a simulation, practical applications are less likely to have a constant mass flow rate fan specified that could maintain a constant free stream velocity. A more realistic scenario is the use of a fan that conforms to a typical inverse relationship between head loss and volumetric flow rate. As the pressure drop across the system increases due a flow obstruction associated with the specification of more fins, the volumetric flow rate and in turn the free stream velocity decreases. As shown in Fig. 4, a ETRI-280DM DC fan

TABLE VI
SINGLE PARAMETER OPTIMIZATION USING A SPECIFIED FAN CURVE

iter.	N	θ_b (°C)	ΔP (mm_{H_2O})	Re
1	15.00	17.96	0.31	451.8
2	19.02	15.51	0.48	313.1
3	20.35	15.35	0.55	278.6
4	20.45	15.35	0.56	276.0
5	20.45	15.35	0.56	276.0

TABLE VII
OPTIMIZED CONDITIONS FOR ALL TEST CASES

Case	N	θ_b (°C)	ΔP (mm_{H_2O})	V_f (m/s)	$\dot{S}_{gen}$ (W/°C)
(i)	28.57	11.51	5.62	2.0	0.00435
(ii)	26.77	9.49	7.02	2.81	0.00402
(iii)	38.14	8.66	5.78	3.28	0.00370
(iv)	24.57	7.21	2.13	1.48	0.00294
(v)	19.07	7.20	1.90	1.21	0.00290
(vi)	20.45	15.35	0.56	0.82	0.00522

having a 40 mm by 40 mm profile exhibits a nonlinear relationship between pressure drop and volumetric flow rate

$$\Delta P = 4.5 - 2.13\dot{\Theta} - 2.53\dot{\Theta}^2 + 2.9\dot{\Theta}^3 - 1.06\dot{\Theta}^4 + 0.133\dot{\Theta}^5 \tag{24}$$

where $\dot{\Theta}$ is the volumetric flow in l/s.

Assuming a fin height of 50 mm and various number of fins, system operating points can be determined, that when correlated provide a relationship between the free stream velocity and the number of fins.

The locus of operating points, as shown in Fig. 4, can be curve fit to obtain an expression for free stream velocity based on the number of fins N as follows:

$$V_f = 1.119 + 6.758 \times 10^{-3}\,N - 1.345 \times 10^{-3}\,N^2 + 1.485 \times 10^{-5}\,N^3. \tag{25}$$

Using the above equation for V_f, the optimal number of fins was determined to be $N \approx 20$, as shown in Table VI.

V. SUMMARY AND RECOMMENDATIONS

A procedure is presented that allows design parameters in a plate fin heat sink to be optimized. The procedure is based on the minimization of entropy generation resulting from viscous fluid effects and heat transfer, both in the cooling medium and within the internal conductive path of the heat sink. The model clearly demonstrates a rapid, stable procedure for obtaining optimum design conditions without resorting to parametric analysis using repeated iterations with a thermal analysis tool.

The results for the six cases presented in the course of demonstrating the feasibility of the model are summarized in Table VII. In each of the first five cases, the overall rate of entropy generation is decreased as additional unconstrained variables are included in the optimization procedure. In theory, the optimization procedure should not constrain any of the relevant design parameters, however, manufacturing practicalities often take precedent over thermal considerations in the design and manufacture of heat sinks.

In the present examples, unconstrained nonlinear optimization methods were applied in each case. Future modifications to the optimization model will allow design variables to be constrained at a predetermined minimum or maximum but otherwise free to go to an optimized value. For instance, the fin height introduced in Cases (i)–(ii) was 25 mm. Extensions to the procedure presented here will allow a maximum height of 25 mm to be imposed, but fin heights of less than 25 mm are feasible if the entropy generation is minimized in these cases. Cases (iv) and (v) clearly indicated a need for this modification as the optimized fin height went to 96.6 and 122 mm, respectively, as the fin height parameter was unconstrained. Other examples of the need for constrained problems include the specification of maximum heat sink operating temperatures which can be directly linked to limitations on junction operating temperatures. In addition, the overall heat sink mass may be a constrained in weight sensitive applications. It is anticipated that the present method may be applied to constrained problems through the use of alternate methods such as Lagrange multipliers [12]. This method increases the number of variables depending upon the number and nature of the imposed constraints.

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